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Kurt Godel's Rebuttal of Formalism in Mathematics

Christopher Alexander Udofia PhD
udofiachris@yahoo.com
Philosophy Department
Akwa Ibom State University
Akwa Ibom State- Nigeria

Abstract: A major cognitive barrier that Philosophers of Mathematics encounter in the course of unraveling mathematical related ideas of thinkers has been that of rendering the sophisticated crypto-codes associated with the mathematical equations of such ideas into a non-technical form for easy comprehension by both experts and non-experts. The ultimate objective of this research with the title, *Kurt Godel's Rebuttal of Formalism in Mathematics*, is to present the incompleteness theorems of Kurt Gödel in a non-technical manner so as to make it equally intelligible to both philosophers and mathematicians. In order to achieve the prime goal stated above, the research will undertake a narrative exposition of some of the attempts to formalize the axioms of mathematics. In the exposition, the contributions of Euclid, Dedekind, Hilbert, Whitehead and Russell in the attempt to build Mathematics into a rigorous formal system that is complete, consistent and decidable would be analyzed. The hallmark of this essay is contained in the discussion of Kurt Gödel's rebuttal of formalism through the instrumentality of his incompleteness theorems.

Key Words: Formalism, Godel, Incompleteness Theorems, Set Theory, Paradox.

Introduction

Gödel's theorem is perceived by many as "the third leg, together with Heisenberg's uncertainty principle and Einstein's relativity, of that tripod of theoretical cataclysms that have been felt to force disturbances deep down in the foundation of the exact sciences" (Goldstein, 21-22). What is popularly referred to as Gödel's theorem is a conjunction of two theorems, propounded by Gödel in his work entitled "On Formally Undecidable Propositions of Principia Mathematica and Related Systems 1." Godel's work was a pessimistic response to David Hilbert's 1900 lecture, where he as a leading mathematician in the modern era set the pace for mathematicians in the Twentieth Century by outlining a set of twenty three problems that mathematicians must solve in the Century. He was so optimistic of the solvability of every mathematical problem that he translated this optimism into the axioms of solvability in his famous no *ignorabimus* statement which states that there are no unsolvable problems in Mathematics.

Among the problems posed by Hilbert was the second problem which demanded for a proof of the consistency of the axioms of mathematics. The first path to the solution of this problem required configuring the entire mathematics into one axiomatic formal system from which one could derive all the theorems of mathematics according to fixed syntactical rules of inference and the second path required demonstrating that the formal system is consistent. Such a formal system that follows the first path and contains all the rules for deciding or proving all mathematical theorems is said to be *complete*. A formal system is said to be *consistent* if it is incapable of generating contradictory propositions and inconsistent if otherwise.

Alfred North Whitehead and Bertrand Russell rose to the task of constructing a formal system of mathematics that bears the features of completeness and consistency in their novel *Principia Mathematical*. Kurt Gödel critiqued this effort of White Head and Russell and by extension, Hilbert's programme, in his ground breaking theorems mentioned above and proved that the formal system of *Principia Mathematica* and other related formal systems cannot fulfill the criteria of completeness and consistency. This work can be said to be a discursive presentation of the Godellian paraphernalia.

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The Remote Euclidean Background to Kurt Gödel's Incompleteness Theorems in Relation to the Axiomatic Formalization of Mathematics

A proper understanding of the Godellian theorem requires a brief historical survey of the axiomatic formalization of mathematics programme which gave rise to Gödel's famous theorems. The notion of an axiomatic system is remotely traceable to Euclid. In his *Elements of Geometry*, which was regarded as a bible of geometry for over two thousand years, Euclid rigorously established the science of geometry on a system of definitions, postulates and axioms. The entire science and theorems of geometry were deduced from the principles embedded in Euclid's elements which were enunciated in the definitions, postulates and axioms. Douglas Hofstadter (88) observes that Euclid set up the paradigm of rigour in mathematics because he so constructed his geometry in such a way that any given theorem or proposition of geometry depended only on or was to be derived from the hitherto established principles and axioms. This implies that every valid principle and proposition of the system must be at realm, consistent or non-contradictory to the axioms. G. Kneebone, *Mathematical Logic and the Foundations of Mathematics: An Introductory Survey* (135-136) spells out some principles of Euclid's axiomatic system classified under the three headings of definitions, postulates and common notions or axioms. The definitions include;

1. A point is that which has no part
2. A line is a breathless length
3. A straight line is a line which lies evenly with the points on itself.

23. Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

There are five postulates restated by Douglas Hofstadter (90) thus;

1. A straight line segment can be drawn joining any two points.
2. Any straight line segment can be extended indefinitely in a straight line.
3. Given any straight line segment, a circle can be drawn having the segment as a radius and one end point as center.
4. All right angles are congruent.
5. If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles then the two lines inevitably must intersect each other on that side if extended far enough.

Finally, the five axioms of Euclid exposed by Kneebone (136) are;

1. Things which are equal to the same thing are also equal to one another.
2. If equals be added to equals, the whole are equal.
3. If equals be subtracted from equals, the remainders are equal.
4. Things which coincide with one another are equal to one another.
5. The whole is greater than the part.

The history of Euclid's principles will be incomplete without the mention of the controversial fifth postulate (the parallel postulate) and how it paved the way for the emergence of non-Euclidean geometry. Ernest Nagel and James Newman (9) observe in respect of the fifth postulate that for some reasons, the postulate did not appear self-consistent to the ancient Greeks. The major reason for its lack of consistency, according to the duo, is that Euclid defines parallel lines as straight lines in a plane, which if produced indefinitely in both directions do not meet. Thus, to say that two lines are parallel is to assert the impossibility of the two lines meeting even at infinity. The fifth postulate appears to be an apparent contradiction of Euclid's definition of the parallel lines. Another controversial assertion of the fifth postulate was the assumption that "through every point P not on a given line L there exists exactly one parallel to L,i.e. one straight line which does not meet L." (Carl G. Hempel "What makes Mathematical Truth Indelible?"464.) Thus, it is evident that a proof of Euclid's parallel postulate on the basis of the other postulates is impossible.

This independence of the fifth postulate from the other four does not however mean that it is false. Alfred Tarski (461) observes that "Euclid's parallel postulate is not false, but it is true only on a plane (two dimensional) surface like a chalkboard" ("Axiomatisation: The Methodology of the Deductive Sciences"461). In the 19th century, the Russian N.I. Lobacheusky and the Hungarian J. Bolyai

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simultaneously but independently discover the non-Euclidean geometry called hyperbolic geometry. Later Bernhard Riemann developed an alternative geometry called elliptical geometry. Hempel explains that many of the theorems of these non- Euclidean geometries are at variance with Euclid thus:

> … in the hyperbolic geometry of two dimensions, there exist, for each straight line L, through any point P not on L, infinitely many straight lines which do not meet L, also, the sum of the angles in any triangle is less than two right angles. In elliptic geometry, this angle sum is always greater than two right angles; no two straight lines are parallel, and while two different points usually determine exactly one straight line connecting them(as they always do in Euclidean geometry), there are certain pairs of points which are connected by infinitely many straight lines. An illustration of this latter type of geometry is provided by the geometrical structure of that curved two- dimensional space which is represented by the surface of a sphere, when the concept of a straight line is interpreted by that of great circle on the sphere. In this space, there are no parallel lines since any two great circles intersect; the endpoints of any diameter of the sphere are points connected by infinitely many different "straight lines," and the sum of the angles in a triangle is always in excess of two right angles. (467)

Fundamentally, Euclid's axiomatisation programme lacked the rigour of the formal language necessary for the eradication and elimination of ambiguity in mathematic. This is borne out of the fact that Euclid's principles were mostly rendered in ordinary natural language which is most times burdened by imprecision, vagueness and equivocation. In forestalling this defect of ordinary language, Francesco Berto asserts that "philosophers have been envisaging artificial, formal languages to serve as antidotes to the deficiencies of natural language, and in which rigorous science could be formulated: languages whose syntax was to be absolutely precise, and whose expressions were to have completely precise and univocal meanings" (16).

Another deficiency of the ancient axiomatic system was that it lacked an abstract conception of number. Kneebone (137) observes that this made mathematicians to conceive magnitude mostly in terms of the geometrical areas, lengths or volumes. These stated deficiencies of the ancient deductive system were remedied efficiently in the modern era without necessarily discarding the entire idea of the axiomatic system. The overriding influence the axiomatic system exerts on thinkers in all era of philosophy is captured by Nagel *et al* thus:

> The axiomatic development of geometry made a powerful impression upon thinkers throughout the ages; for the relatively small numbers of axioms carry the whole weight of the inexhaustible numerous propositions derivable form them. Moreover, if in some way the truth of the axioms can be established… both the truth and the mutual consistency of all the theorems are automatically guaranteed. For these reasons the axiomatic form of geometry appeared to many generations of outstanding thinkers as the model of scientific knowledge at its best. It was natural to ask, therefore, whether other branches of thought besides geometry can be placed upon a secure axiomatic foundation. (3)

The Proximate Modern Background to Kurt Gödel's Incompleteness Theorems in Relation to the Axiomatic Formalization of Mathematics

The admiration of the axiomatic system came with the adoption and exportation of this system to all the branches of mathematics. However, in the modern era, the rigour of a formal language was added to the axiomatic system to make it devoid of the ambiguities of the natural language. Thus, in the modern era of mathematics, many attempts were made to eliminate contradictions and inconsistencies in mathematics by reducing all mathematical expressions to rigorous symbols, signs and formulae. A deductive system that is so rigorised is called a formal system. One of the early masters who attempted an exportation of the axiomatic paradigm to Arithmetics or number theory was Richard Dedekind. He was so appalled by the lack of a rigorous basis for arithmetic that he had to constantly make recourse to axiomatic geometrical intuitions as an indispensable didactic tool for his lectures on differential calculus, when he was appointed as a Professor at Zurich ("Dedekind, Continuity and Irrational Numbers" 767). He therefore attempted to elaborate an abstract basis for the rigorous foundation of mathematics. This abstract foundation, for

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Dedekind is logic and not intuition, as can be seen below, in his conception of number. Dedekind holds the concept of number to be an immediate outcome of the laws of thought devoid of intuition. In respect of this, he said;

> "In calling arithmetic (algebra, analysis) only as a part of logic, I am already asserting that I hold the concept of number to be wholly independent of representations or intuitions of space and time and that I hold it rather to be a product of the pure laws of thought… if we scrutinize closely what is done in counting a set or number things, we are led to consider the ability of the mind to relate things to things, to let a thing correspond to a thing, or to represented a thing by a thing, an ability without which no thinking is possible ("Was Sind und was Sollen die Zahlen" or "On what Numbers ought to be").

Here, Dedekind is clearly articulating the importance of founding mathematics on the principles of relation between things which in this case implied logic. In this respect, one other seminal mind who attempted establishing mathematics on a rigorous basis by symbolically axiomatizing the concept of number, which was simply conceived as intuitive and undefined by the Greeks, was Guiseppe Peano. Peano carried out this task in his "The Principles of Arithmetic, Presented by a New Method." In this work, Peano (94) stated the axioms of arithmetic of which the five most celebrated ones are;

- Principle one (P1) Zero is a number.
- Principle two (P2) The successor of any number is a number.
- Principle three (P3) Zero is not the successor of any number.
- Principle four (P4) Any two numbers with the same successor are the same number.
- Principle five (P5) Any property of zero that is also a property of the successor of any number having it is a property of all numbers.

Though Peano rendered these axioms in symbols, some logicians criticized his work as not adequately rigorous. Francesco observes against Peano that "Peano's proofs were rather informal, and the task of establishing the correctness of the deductive passages was often simple left of the reader" (32). Apart from this defect, Francesco (29) observes that Frege and Russell in their logicist enterprise demanded that numbers should not simply be conceived as primitive and intuitive, as Peano did in his notion of natural numbers, but should be definable in terms of sets, and the properties of relations between sets. This disposition is captured in Russell's (*Principle* 106) definition of mathematics as the class of propositions which assert formal implications and contain logical constants.

Without prejudice to the numerous great philosophers and mathematicians and their monumental achievements, the most outstanding great minds of immediate relevance to this research whose works and ideas in the construction of a formal system constituted the immediate and proximate background and impetus to Godel's incompleteness theorems are Thomas Hilbert and Bertrand Russell. Solomon Feferman (*In the Light of Logic* 3) and Francesco Berto in his *There is Something about Godel* (39-40) expose a scintillating profile of Hilbert as a Superstar in mathematics who is ranked alongside Henri Poincare, as one of the most seminal minds and most influential mathematics character of the twentieth century era.

In 1900, Hilbert gave one of the most profound lectures at the Second International Congress of Mathematics held in Paris. His lecture which was titled "Mathematical Problems" contained a list of twenty three (23) problems. The list became so famous as a determinant of the scope of the major preoccupations and tasks of mathematicians in the 20th century such that outstanding mathematicians who aim at the Field Medal, equivalent of the Oscar, must address one of the unsolved problems in the list. Hilbert was so convinced of the possibility of deriving a solution to all mathematical problems to the extent that he premised his list of the problems in mathematics with the popular axiom of solvability, thus;

> Is this axiom of the solvability of every problem a peculiarity characteristic of mathematical thought alone, or is it possibly a general law inherent in the nature of the mind, that all questions which it asks must be answerable? …. This conviction of the solvability of every mathematical problem is a powerful incentive to the worker. We hear within us the perpetual call: there is the problem. Seek its solution. You can find it by pure reason, for in mathematics *there is no ignorabimus (Hilbert,*"Problem of Mathematics" in Solomon Feferman's *In the Light of Logic* 21).

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Feferman (4) has cynically observed against the famous axiom of solvability of every problem in mathematics that it was daring to assume that the power of human thought is limitless. Aside this, one of the problems posed by Hilbert, which is of immediate concern to this work, is the second problem which demanded for a proof that the axioms of arithmetic are compatible or consistent. Hilbert articulated this problem thus: "But above all I wish to designate the following as the most important among the numerous questions which can be asked with regard to axioms. To prove that they are not contradictory, that is, that a definite number of logical steps based upon them never lead to contradictory results ("Problems of Mathematics" 21-22).

The Conundrum of Set Theoretic Paradox, the Advances of Hilbert, Whitehead and Russell and the Rebuttal from Kurt Godel's Incompleteness Theorems

The problem of the consistency of arithmetic arose when the early masters of logic and mathematics attempted an exportation of the axiomatic paradigm of geometry to arithmetic or number theory which is also called set theory so as to render arithmetic bereft of the ambiguities of the natural language and enhance an abstract conception of number which Euclid's postulates lacked. Thus Jose Ferreiros (292) observes that Cantor defines "set" as any collection into a whole, M, of definite, distinguishable objects m (which will be called 'elements' of M) of our intuition or thought. Thus, the term 'set' is synonymous with 'aggregate' 'collection' or 'class'. However, one common peculiarity of every set, class, or collection is that it contains a number of elements. In this wise Russell defines number in terms of set or class thus; "mathematically, a number is nothing but a class of similar classes" (*Principles of Mathematics*116.).

The observation above is further corroborated by Francesco Berto thus;

> One of the features that renders set theory important for mathematics is the fact that sets can be members or elements of sets in their turn. In this sense, sets are not just collections of objects, but objects themselves… Frege and Russell based their logicist approach on the possibility of reducing numbers to sets by considering them as sets of sets (19).

This idea that sets can be members of themselves ignited the crisis of paradoxes in set theory- which was the foundation of mathematics. Two of some of the most profound minds in mathematical logic who advocated that mathematics should be based on or reduced to or founded on axiomatised set theory were Gottlob Frege and Russell. Though the latter is of immediate relevance to this work, it is worth mentioning that the duo independently formalized the elaborate programmes for the reduction of mathematics to logic. The former accomplished that project in his *Begriffsschrift* (*Concept-Script*) while the later achieved it in his Magnus opus *The Principia Mathematica.*

The discovery of the set theoretical paradox that constituted a conundrum that rocked the attempt at founding mathematics on a formalized axiomatic set theory is credited to Russell. This discovery was made in the course of Russell's study of Cantor's and Frege's works and he communicated his discovery of a contradiction in set theory to Frege thus;

> You state that a function, too, can act as the indeterminate element. This I formerly believed, but now this view seems doubtful to me because of the following contradiction. Let w be the predicate: to be a predicate that cannot be predicated of itself, can w be predicted of itself. From each answer its opposite follows ("Letter to Frege" 124-125).

The discovery of this contradiction in the foundation of mathematics constituted a catastrophe to Frege's *Grundgesetze der Arithmetic (The Ground Work of Arithmetic*) and Frege responded thus: "your discovery of the contradiction caused me the greatest surprise and, I would almost say, consternation, since it has shaken the basis on which I intended to build arithmetic" ("Letter to Russell" 127).

A statement of Russell's paradox (*Principle* 107) asserts that the class of all classes which are not members of themselves can be proved to be and not to be members of itself. Interpreting class as set, Abraham Fraenkel et al (*Foundations of Set Theory* 5) in proof of this Russell's antinomy, explains that though certain sets can be clearly shown not to be members of themselves, like the set of planets which obviously is not a planet (not a member of itself) but there is a set of which it cannot be definitely determined whether it is a member of itself or not. An instance of this is: the set of all sets that are not members of themselves. Denoting this set as 'S', he observes that if one takes 'S' as a member of itself, then it belongs to the set of all sets that are not members of themselves, therefore, it is not a member of itself. If

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one takes 'S' as not a member of 'S' then it does not belong to the set of all sets that are not members of themselves, therefore, it is a member of itself. Thus 'S' is a member of 'S' if and only if 'S' is not a member of 'S'. This is evidently a contradiction. Detailing how Russell came to the conception of this contradiction in set theory, it is explained in John Slater's introductory note to *The Principles* (xxviii) that Russell came to conceive the idea of the contradiction in set theory when he noticed that some classes are members of themselves, like the class of abstract ideas, which is also an abstract idea, whereas others are not, like the class of bicycles, is not a bicycle. These classes which are not members of themselves, Russell regarded them as "ordinary" classes. Using O to designate all the classes which are not members of themselves (the ordinary classes), Russell then asked whether O was a member of itself or not. Suppose that O is a member of O, then since all members of O are non-self- membered, it follows that O is not a member of O. This paradox is a vivid violation of some logical principles like the law of Non-contradiction, the Principle of excluded middle etc.

Evaluating the nature of paradoxes, Russell observes that all logical contradictions have the character of circularity otherwise called self-referentiality which he gave the name, "reflexiveness". Explaining this, he says in his *Principia Mathematica* that; "In all the above contradiction… there is a common characteristic, which we may describe as self- reference or reflexiveness. The remark of Epimenides must include itself in its own scope. If all classes, provided there are not members of themselves, are members of [R], this must also apply to [R]" (61-62).

This vicious circularity or self- reflexiveness is also a characteristic of the Liar's and the Barber's paradoxes. The devastating impact that the set theoretic paradox had on the foundation of mathematics is here expressed by Hilbert thus;

> Let us admit that the situation in which we presently find ourselves with respect to the paradoxes is in the long run intolerable. Just think: in mathematics, this paragon of reliability and truth, the very notions and inferences, as everyone learns, teaches and uses them, lead to absurdities. And where else would reliability and truth be found if even mathematical thinking fails *("On the Infinite" 374)*

Convinced that there are no unsolvable problems in mathematics or no ignorabimus (the non ignorabimus of Hilbert is in reference to an old saying, "ignoramus et ignorabimus, meaning, we do not know and we shall never know) he sets out to rid mathematics of inconsistencies or contradictions and thus solve the problem of the crisis at the foundation of mathematics through a progrmmme called formalism. This formal system possesses characteristics of the classical axiomatic method in that it is made up of clearly defined terms and axioms, however, among the deficiencies of Euclid's axiomatic method, according to Torkel Franzen (17) are that the language of the system is not formally specified that is, it is vague and natural, and that his proofs use geometrical assumptions not contained in the postulates. Thus, the remarkable difference between the classical axiomatic system and the formal axiomatic system is that the theories of the formal system have been completely translated into a rigorous artificial language of symbolic logic devoid of any extraneous meaning apart from the one specified in the system.

Exposing his notion of a formal system and its components, Hilbert asserts that:

> … We now divests the logical signs of all meaning just as we did the mathematical ones, declare that the formulas of the logical calculus do not mean anything in themselves….In a way that actually corresponds to the transition from contextual number theory to formal algebra. We regard the signs and operation symbols of the logical calculus as detached from their contextual meaning. In this way we now finally obtain, in place of the contextual mathematical science that is communicated by means of ordinary language, an inventory of formulas that are formed from mathematical and logical signs and follow each other according to definite rules. Certain of these formulas correspond to the mathematical axioms and to contextual inference there corresponds the rules according to which the formulas follow each other, hence contextual inference is replaced by manipulation of signs according to rules, and in this way the full transition from naive to a formal treatment is now accomplished (Hilbert, "On the Infinite" 381)

Hilbert therefore conceived a formalized axiomatic system as a system individuated by signs made up of primitive symbols of logic, formulas, and rules of inference, formal proofs, and theorems. Formalizing an axiomatized system therefore, requires choosing an artificial language for the theory. Such an artificial language for Hilbert is made up of variables like A,B,C, to Z, logical symbols like → for "if then", ↔ for "if and only if", ~ for "not", Λ for "and ", ν for "or," the equality signs " =," and the two quantifiers; the "for all" quantifier ∀ and "there exist" quantifier ∃ and finally the undefined primitive terms of arithmetic and their appropriate symbols namely "zero" "addition" and "multiplication" symbolized as 0,+, x.

Gregory Chaitin (*Thinking about Gödel and Turring: Essays on Complexity*77-79) observes that Hilbert's formal system was demanded to possess three cardinal properties:

1. It must be complete.
2. It must be consistent.
3. It must be decidable.

The first condition means that every statement in that system should be proved. The second requires that if a system is proved true, it cannot at the same time be proved false. The third implies that there exists a method or an algorithm that is guaranteed to prove the statement either true or false. A system that possesses within itself demonstrable rules of proof is said to be complete and decidable while the one that lacks it is said to be incomplete and undecidable. A system that does not allow for contradictions is said to be consistent while the one that allows is said to be inconsistent.

It was Hilbert's cherished aspiration that once such a formal theory that possesses those characteristics is constructed then contradictions will be banished from the enterprise of mathematics and all the problems of mathematics would be decided.

In response to Hilbert's call for the solution to the consistency problem and the formal axiomatization of mathematics, Bertrand Russell and Alfred North Whitehead came out with their Magnum Opus, *The Principia Mathematica*. The *Principia* was a three volume work in which all the methods of proof used in mathematics are reduced to a few axioms and rules of inference. The rationale for this grand formal system was that if the axioms of number theory are expressed as derivable as theorems of formal logic, then the question of the consistency of mathematics is solved when the consistency of the axioms of logic is demonstrated. This rationale is an emanation from the Frege - Russell thesis that mathematics is reducible to logic. Russell introduced the theory of logical types in the *Principia* to tame and outflank the antinomial Paradoxes. In this respect, Russell developed:

> A rapid hierarchy of types of objects: individuals, sets, sets of sets, sets of sets of sets……. What belongs to a certain logical type can be (or not be) a member only of what belongs to the immediately superior logical type. The membership relation can hold, or fail to hold only between an individual and a set of individuals or between a set of individuals and a set of individuals; and so on. The construction allows any sets to contain only things of one order; it allows only set composed so to speak, of objects that are homogenous with respect to the hierarchy. Therefore, there is no set of all sets or of all ordinals etc. (Francesco Berto, *There is Something about Godel*37).

Russell's *Principia* came as a most welcome response to the search for a consistent and complete formal system of mathematics. It was at this critical stage where the crisis of inconsistency instigated by the paradoxes seems to have been settled, that Kurt Gödel came out with a rebuttal of Hilbert's non ignorabimus and the consistency and completeness of the *Principia Mathematica*. Kurt Gödel in his 1931 paper "On Formally Undecidable Propositions of Principia Mathematical and Related Systems" sets forth two fundamental theorems that would serve to foil Hilbert's optimism about a formal consistent and complete system of mathematics. Kurt Gödel offers a brief background overview of the emergence of formalism thus:

> The development of mathematics towards greater precision has led, as is well known, to the formalization of large tracts of it, so that one can prove any theorem using nothing but a few mathematical rules. The most comprehensive formal systems that have been are the system of *Principia Mathematica* (PM)on the one hand and the Zermelo-Fraenkel axiom of set theory... on the other. These two systems are so comprehensive that in them all methods of proofs used in mathematics today are formalized, that is, reduced to a few axioms and rules of inference. One might therefore conjecture that these axioms and rules of inference are sufficient to decide any mathematical axiom that can at all be formally be expressed in these

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> systems. It will be shown below that this is not the case, that on the contrary there are in the two systems mentioned relatively simple problems in the theory of integer that cannot be decided on the basis of the axioms. This situation... holds for a wide class of formal systems. (“On Formally Undecidable Propositions of Principia Mathematica and Related Systems 1, 145")

In his statement of the first incompleteness theorem, Gödel asserts that: “Every formal system with finitely many axioms that contains the arithmetic of the natural number is incomplete” (“On Undecidable Sentences,” *Kurt Gödel Collected Works. Volume 111: Unpublished Essays and Lectures* 32).His statement of the second incompleteness theorem reads; “The consistency of a formal system can never be established by methods of proof…formalized in the system in question; rather, for that one always needs some methods of proof that transcend the system"(“On Undecidable Sentences”35).

Without prejudice to the technical sophistications of Gödel’s proof, he (“On formally Undecidable Propositions of Principia Mathematica and Related Systems 1”, 145-149) offered the Liar’s paradox as a heuristic analogy for comprehending the import of his theorem.

The Liar’s paradox is credited to Epimenides; a Greek philosopher and prophet, and its form is reflected in the self-reflexive statement which says, “This very statement is false”. The Biblical counterpart of this paradox is exemplified in the Pauline passage thus: “one of themselves, even a prophet of their own said, “Cretians are always liars… this witness is true (Titus1:12-13). These statements share the features of circular self reflexivity and self contradiction. It is self reflexive because it is making assertion about itself and contradictory because if one considers the statement as true, it contradicts what the statement asserts, namely, that it is false. On the other hand, if one appraises it as false, then it corresponds with the true assertion of the statement, namely, that it is false. Either way, the statement is inconsistent and contradictory because it is both true and false simultaneously. One most profound and disturbing implications of paradoxical statements like the statement above is that if they could be introduced into a formal system, they would render such a system vulnerable to incompleteness, inconsistency and undecidability. This was exactly what Godel did to the formal system of *Principia Mathematica* to proof that it is incomplete and inconsistent. Since a formal system of mathematic is a system of axioms and mathematics deal with provability/unprovability of theorems, Godel had to device a way of constructing an axiomatic mathematical proposition about the system of *Principia Mathematica* that makes a self- reflexive mathematical assertion resembling the Liar’s paradox. The self- referential proposition that Godel made was: *G is unprovable in this system*. If it is true that this statement is unprovable in the formal system of *Principia Mathematica*, then it implies that there exist true statements in the system that cannot be proved and by this demonstration, the system of *Principia Mathematica* has failed to meet the first cardinal criterion of a formal system, that of completeness, which requires that all the propositions in the system must be proved. With this, Godel, proved his first incompleteness theorem that a formal system contains true but unprovable propositions.

Next, if the proposition that, *G is unprovable in this system,* is false, that means that (it is true that) G is provable and this results in a situation where the proposition is both false and true simultaneously, hence a contradiction or inconsistency is generated in the formal system. Through this result, the system fails the second criterion of consistency which Hilbert wanted to achieve. With this demonstration, Gödel proved his second incompleteness theorem that a formal system that can be proved as complete cannot at the same time demonstrate its consistency. Finally, since the consistency of the system cannot be demonstrated within the system, then the system fails the criterion of decidability.

Conclusion

This research exercise was propelled by the prime aim of rendering the incompleteness of Kurt Godel in a manner that will be meaningfully intelligible to both experts and non-experts by presenting those theorems bereft of the complex equation coding usually associated with such mathematical ideas. Through a narrative discourse of the remote and proximate precursors who contributed immensely to the idea of axiomatic formalization of mathematics, like Euclid, Dedekind, Hilbert, Whitehead and Russell, the work discussed how Gödel employed the incompleteness theorems to refute the formalization program of Hilbert, Whitehead and Russell.

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