Analysis of some problems about the sum of digits of a positive integers

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Abstract: This article presents analysis of some problems about the sum of digits of a positive integers. Problems about the sum of digits of a positive integer often occur in mathematical contests because of their difficulty and the lack of standard ways to tackle the problem. Therefore, in such cases, a analysis of common methods is useful.

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Problems about the sum of digits of a positive integer often occur in mathematical contests because of their difficulty and the lack of standard ways to tackle the problem. This is why a synthesis of the most frequent techniques that occur in such cases would be useful. We have selected several representative problems to show how the main results and techniques work and why they are so important.

We will work only in base 10 and we will denote the decimal sum of digits of the positive integer x by $s(x)$. The following ”formula” can be checked easily:

$$s(n) = n - 9\sum_{k\geq 1}\left[\frac{n}{10^k}\right] \qquad (1)$$

From (1) we can easily deduce some well-known results about $s(n)$ such as $s(n) \equiv n(mod\ 9)\ and\ s(m + n) \leq s(m) + s(n)$. Unfortunately, (1) is a clumsy formula, which can hardly be used in applications. On the other hand, there are several more or less known results about sum of digits, results which may offer simple ways to tackle hard problems. This is what we will discuss about in the following.

The easiest of these techniques is, probably, just the careful analysis of the structure of the numbers and their digits. This can work surprisingly well, as we will see in the following examples.

Problem 1. Prove that among any 79 consecutive numbers, one can choose at least one whose sum of digits is a multiple of 13.

Solution. Note that among the fifirst 40 numbers, there are exactly 4 multiples of 10. Also, it is clear that the last but one digit of one of them is at least 6. Let x be this number. Obviously, $x, x + 1, \ldots, x + 39$ are among our numbers, so $s(x), s(x) + 1, \ldots, s(x) + 12$ occur as sum of digits in some of our numbers. Obviously, one of these numbers is a multiple of 13 and we are done.

We will continue with two harder problems, which still do not require any special result or technique.

Problem 2. Find the greatest N such that one can find N consecutive numbers with the property that the sum of digits of the k-th number is divisible by k, for $k = 1, 2, \ldots, N$.

Solution. The answer here is not trivial at all, namely 21. The main idea is that among $s(n + 2), s(n + 12)$ and $s(n + 22)$ there are two consecutive numbers, which is impossible since they should all be even. In truth, we make transports at $a + 10$ only when the last but one digit of a is 9, but this situation can occur at most once in our case. So, for $N > 21$, we have no solution. For $N = 21$, we can choose $N + 1, N + 2, \ldots, N + 21$, where $N = 291 \cdot 1011! - 12$. For $i = 1$ we have nothing to prove.

For

$$2 \leq i \leq 11, s(N + i) = 2 + 9 + 0 + 9(11! - 1) + i - 2 = i + 11!$$

while for

$$12 \leq i \leq 21, s(N + i) = 2 + 9 + 1 + (i - 12) = i,$$

so our numbers have the desired property.

Problem 3. How many positive integers $n \leq 102005$ can be written as the sum of 2 positive integers with the same sum of digits?

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Solution. Answer:$10^{2005} - 9023$. At first glance, it might seem almost impossible to find the exact number of positive integers with this property. In fact, the following is true: a positive integer cannot be written as the sum of two numbers with the same sum of digits iff all of its digits (eventually) excepting the first are 9 and the sum of its digits is odd.

Let n be such a number. Suppose there are $a, b \in Z+$ such that $n = a + b$ and $s(a) = s(b)$. The main fact is that when we add $a + b = n$, there are no transports. This is clear enough. It follows that $s(n) = s(a) + s(b) = 2s(a)$, which is impossible since $s(n)$ is odd. Now we will prove that any number n which is not one of the numbers stated above, can be written as the sum of 2 positive integers with the same sum of digits. We will start with the following:

Lemma. *There is* $a \le n$ such that

$s(a) \equiv s(n - a)(mod2)$.

Proof. If the $s(n)$ is even, take $a = 0$. If $s(n)$ is odd, then n must have a digit which is not the first and is not equal to 9, otherwise it would have one of the forbidden forms. Let c be the value of this digit and p its position (from right to left). Then let us chose $a = 10p - 1(c + 1)$. At the adding $a + (n - a) = n$ there is exactly one transport, so

$s(a) + s(n - a) = 9 + s(n) \equiv 0(mod2) \Rightarrow s(a) \equiv s(n - a)(mod2)$ which proves our claim. Back to the original problem. All we have to do now is take one-by one a ”unity” from a number and give it to the other until the 2 numbers have the same sum of digits. This will happen since they have the same parity. So, let us do this rigorously. Let

$a = \overline{a_1 a_2 \ldots a_k}, n - a = \overline{b_1 b_2 \ldots b_k}$

The lemma shows that the number of elements of the set $I = \{i \in \{1, 2, \ldots, k\}$: 2 does not divide $\{a_i + b_i\}$ is even, so it can be divided into 2 sets with the same number of elements, say I_1 and I_2. For $i = 1, 2, \ldots, k$ define

$A_i = \frac{a_i + b_i}{2}$ $if\ i \in I_1, \frac{a_i + b_i + 1}{2}$ $if\ i \in I_1 or \frac{a_i + b_i - 1}{2}$ $if\ i \in I_2$

and $B_i = a_i + b_i - A_i$. It is clear that the numbers

$$A = \overline{A_1 A_2 \ldots A_k}, \qquad B = \overline{B_1 B_2 \ldots B_k}$$

have the property that $s(A) = s(B)$ and $A + B = n$. The proof is complete. We have previously seen that $s(n) \equiv n(mod\ 9)$. This is probably the most famous property of the function *s* and it has a series of remarkable applications. Sometimes it is combined with some simple inequalities such as $s(n) \le 9(b\ lg\ nc + 1)$. Some immediate applications are the following:

Problem 4. Find all *n* for which one can find *a* and *b* such that

$$s(a) = s(b) = s(a + b) = n.$$

Solution. We have $a \equiv b \equiv a + b \equiv n(mod9)$, so 9 divides n. If $n = 9k$, we can take $a = b = 10k - 1$ and we are done since

$s(10^k - 1) = s(2 \cdot 10^k - 2) = 9^k$.

Problem5. Find all the possible values of the sum of digits of a perfect square.

Solution. What does sum of digits has to do with perfect squares? Apparently, nothing, but perfect squares do have something to do with remainders mod 9! In fact, it is very easy to prove that the only possible values of a perfect square mod 9 are $0, 1, 4$ and 7. So, we deduce that the sum of digits of a perfect square must be congruent to $0, 1, 4\ or\ 7\ mod\ 9$. To prove that all such numbers work, we will use a small and very common (but worth to remember!) trick: use numbers that consist almost only of 9-s. We have the following identities:

$$\underbrace{99\ldots99}_{n}{}^2 = \underbrace{99\ldots99}_{n-1}8\underbrace{00\ldots001}_{n-1} \Rightarrow s\left(\underbrace{99\ldots99}_{n}{}^2\right) = 9n$$

$$\underbrace{99\ldots99}_{n-1}1^2 = \underbrace{99\ldots99}_{n-2}82\underbrace{00\ldots00}_{n-2}81 \Rightarrow s\left(\underbrace{99\ldots99}_{n-1}1^2\right) = 9n + 1$$

$$\underbrace{99\ldots99}_{n-1}2^2 = \underbrace{99\ldots99}_{n-2}84\underbrace{00\ldots00}_{n-2}64 \Rightarrow s\left(\underbrace{99\ldots99}_{n-1}2^2\right) = 9n + 4$$

$$\underbrace{99\ldots99}_{n-1}4^2=\underbrace{99\ldots99}_{n-2}88\underbrace{00\ldots00}_{n-2}36\Rightarrow s\left(\underbrace{99\ldots99}_{n-1}4^2\right)=9n+7$$

and since $s(0) = 0, s(1) = 1, s(4) = 4$ *and* $s(25) = 7$ the proof is complete.

Problem 6. Compute $s\left(s\left(s(4444^{4444})\right)\right)$.

Solution. Using the inequality $s(n) \le 9([\lg n] + 1)$ several times we have

$$s(4444^{4444}) \le 9([\lg 4444^{4444}] + 1) < 9 \cdot 20,000 = 180,000;$$
$$s\left(s(4444^{4444})\right) \le 9([\lg s(4444^{4444})] + 1) \le 9(\lg 180,000 + 1) \le 36$$

So

$$s(s(s(4444^{4444}))) \le 12$$

On the only possible answer is 7.

Finally, we present a beautiful problem which appeared in the Russian Olympiad and, later, in *Kvant*.

Problem 7. Prove that for any *N* there is $n \ge N$ such that $s(3^n) \ge s(3^{n+1})$.

Solution. Suppose by way of contradiction that there is one *N* such that

$s(3^{n+1}) - s(3^n) > 0, \forall n \ge N.$

But, for

$n \ge 2, s(3^{n+1}) - s(3^n) \equiv 0(mod 9),$

So

$s(s(3^{n+1}) \ge 9(n - N), n \ge N + 1.) - s() \ge 9, \forall n \ge N.$

It follows that

$$\sum_{k=N+1}^{n}(3^{k+1} - 3^k) \ge 9(n - N) \Rightarrow s(3^{n+1}) \ge 9(n - N), n \ge N + 1.$$

But

$$s(3^{n+1}) \le 9([\lg 3^{n+1}] + 1),$$

So

$$9n - 9N \le 9 + 9(n + 1)\, lg\, 3,$$

for all $n \ge N + 1$. This is obviously a contradiction. If so far we have studied some remarkable properties of the function *s*, which were quite well-known, it is time to present some problems and results which are less familiar, but interesting and hard. The first result is the following:

Statement. If $1 \le x \le 10^n$, then

$s(x(10^n - 1)) = 9^n.$

Proof. The idea is very simple. All we have to do is write

$$x = \overline{a_1 a_2 \ldots a_j}$$

With $a_j \neq 0$ (we can ignore the final 0-s of (x) and note that

$$x(10^n - 1) = \overline{a_1 a_2 \ldots (a_j - 1) 99 \ldots 99 (9 - a_j)(10 - a_j)},$$

which obviously has the sum of digits equal to 9^n.

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