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Task Cauchy and Carleman Function

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Abstract: In this paper we discuss the continuation polyharmonic function its values and the values of its normal derivative on the smooth side of S the boundary of the infinite D. Using this integral representation, we obtain some properties of the polyharmonic functions of this class.

Key words: Cauchy problem, Carleman function, polyharmonic functions, partial derivatives, normal derivatives.

The paper proposes an explicit continuation formula for solving the Cauchy problem for the polyharmonic equation in the statement of M.M. Lavrentieva. The continuation formulas found here are complete analogues of the classical Riemann, Voltaire and Hadamard formulas that they constructed to solve the Cauchy problem in the theory of linear equations of the second order. Sh. Yarmukhamedov in 2003 in the article "The Cauchy Problem for the Polyharmonic Equation" having solved the problem, obtained the result when the region is simply connected with the boundary -∂D_ρ, consisting of a cone surface. Juraeva N.Yu. 2004 in the article "The Cauchy Problem for Polyharmonic Functions" [3], proved some theorems when D- unlimited area lying in the layer $\{y: y=(y_1, y_2,..., y_m), (y_1, y_2,..., y_m)\in R^m\ , 0<y_m<h\}$ with border $\partial D = L\cup S$, $L=\{y: y_m=0\}$, $S=\{y: y_m = f(y_1,..., y_{m-1})\}$ Where $f(y_1,..., y_{m-1})$ has first-order bounded partial derivatives.

In this paper, similar results are obtained in the case when the region has the following form:

Let be R^m - m- dimensional real Euclidean space, $x=(x_1, x_2, x_3,..., x_m)$, $y=(y_1, y_2, y_3,..., y_m)$, $x\in R^m, y\in R^m$ $x'=(0, x_2,..., x_m)$, $y'=(0, y_2,..., y_m)$, $r^2=|x-y|$, $s=|x'-y'|^2$, $h=\frac{\pi}{\rho}$, $\rho>0$, $\alpha^2=s$

D- unlimited area lying in the layer $\{y: y=(y_1, y_2,..., y_m)\in R, 0<y_m<h\}$ with border $\partial D=\{y: y=(y_1, y_2,..., y_m), y_1=0\}\cup S$, $S=\{y: y=(y_1,..., y_m), y_m=f(y_1,..., y_{m-1})\}$

где $f(y_1,..., y_{m-1})$ the function satisfies the Lyapunov condition with a fixed constant.

The following problem is solved (Cauchy problem).Let me $u\in C^{2n}(D)$ и $\Delta^n u(y)=0,\ y\in D$ (1)

$$u(y)=F_0(y),\quad \Delta u(y)=F_1\ (y),..., \Delta^{n-1}u(y)=F_{n-1}(y),\quad y\in S$$

$$\frac{du(y)}{d\,\bar n}=G_0(y),\quad \frac{d\Delta u(y)}{d\,\bar n}=G_1(y),..., \frac{d\Delta^{n-1}u(y)}{d\,\bar n}=G_{n-1}(y),\quad y\in S, \qquad (2)$$

where $F_i(y), G_i(y)$ given on ∂D continuous functions, $\bar n$ - external normal to ∂D. Restore required $u(y)$ в D.

We assume that the solution $u(y)$ tasks $(1)-(2)$ exists and is continuously differentiable, $2n-1$ times up to the end points of the boundary and satisfies a certain growth condition (correctness class), which ensures the uniqueness of the solution. Then an explicit continuation formula is established, which is a multidimensional analogue of the classical Carleman formula from the theory of analytic functions.

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Functions $\varphi_\sigma(y,x)$ и $\Phi_\sigma(y,x)$ at $s>0,\sigma\geq 0$ we define the following equalities: if $m=2k, k=2,3,...$ то $(-1)^{k-1}(k-2)!\varphi_\sigma(y,x)K(x_m)=\frac{d^{k-2}}{ds^{k-2}}\operatorname{Im}\left[\frac{K(\omega)}{\sqrt{s}(\omega-x_1)}\right], \quad \omega=i\sqrt{s}+y_1$

if $m=2k+1, k=2,3,...$ то

$$(-1)^{k-1}2^{-k}(k-2)!!\varphi_\sigma(y,x)K(x_m)=\frac{d^{k-1}}{ds^{k-1}}\operatorname{Im}\int_0^\infty\left[\frac{K(\omega)}{\omega-x_1}\right]\frac{du}{\sqrt{u^2+s}}, \quad \omega=i\sqrt{s+u^2}+y_1$$

With all the odd $m\geq 3$, as well as even m с condition $2n<m$, we believe

$$\Phi_\sigma(y,x)=C_{n,m}r^{2(n-1)}\varphi_\sigma(y,x), \quad C_{n,m}=(-1)^{n-1}(\Gamma(\frac{m}{2}-n)2^{2n}\pi^{\frac{m}{2}}\Gamma(n))^{-1}$$

For all even m, $m=2k$, $k=1,2,...$ with the condition $2n\geq m$ we believe

$$\Phi_\sigma(y,x)=C_{n,m}\int_0^\infty\operatorname{Im}\left[\frac{K(\omega)}{\omega-x_1}\right](u^2-s)^{n-k}du \quad \omega=iu+y_1 \text{ where}$$

$C_{n,m}=(-1)^{\frac{m}{2}-1}\left(\Gamma(n)\Gamma\left(n-\frac{m}{2}+1\right)2^{2n}\pi^{\frac{m}{2}}\Gamma(n)\right)^{-1}$ and function $K(\omega)$ has the form

$$K(\omega)=\frac{\exp(\sigma\omega-achi\rho_1(\omega-h/2))}{(\omega+x_m+3h)^{n+1}} \quad m=2n+1,\ n\geq 1 \quad K(\omega)=\frac{\exp(\sigma\omega-achi\rho_1(\omega-h/2))}{(\omega+x_m+3h)^{n}} \quad m=2n,\ n\geq 2,$$

Theorem 1. For function $\varphi_\sigma(y,x)$ occurs inequality

$$|\varphi_\sigma(y,x)|\leq\begin{cases}C\sigma^{n-2}\alpha^{-m}\exp(\sigma y_m-a\cos\rho_1 y_m ch\rho_1\alpha), & \alpha\geq 1\\ C\sigma^{n-2}(r^{-m+2}+\alpha^{-1}r^{-m+3}+\sum_{p=1}^{n-2}\alpha^{-2p}r^{-2(n-p-1)}\exp(\sigma y_m-a\cos\rho_1 y_m ch\rho_1\alpha), & 0<\alpha\leq 1\end{cases}$$

Lemma -1. If $\varphi_\sigma(y,x)$ harmonic function в R^m by variable x including point y, *then*

$$\Delta r^k\varphi_\sigma(y,x)=r^{k-2}\varphi_{\sigma,1}(y,x),$$

equality is true where

$$\varphi_{\sigma,1}(y,x)=\sum_{j=1}^{n}(x_j-y_j)\frac{\partial\varphi_\sigma(y,x)}{\partial x_j}+\varphi_\sigma(y,x)$$

function is also a harmonic function в R^m by variable x including point y.

$$\frac{\partial r^k\varphi_\sigma(y,x)}{\partial x_j}=\frac{\partial r^k}{\partial x_j}\varphi_\sigma(y,x)+r^k\frac{\partial\varphi_\sigma(y,x)}{\partial x_j}=k(x_j-y_j)r^{k-2}\varphi_\sigma(y,x)+r^k\frac{\partial\varphi_\sigma(y,x)}{\partial x_j}.$$

Evidence: $\frac{\partial^2 r^k\varphi_\sigma(y,x)}{\partial x_{j_i}^2}=kr^{k-2}\varphi_\sigma(y,x)+k(x_j-y_j)\frac{\partial r^{k-2}}{\partial x_j}\varphi_\sigma(y,x)+k(x_j-y_j)r^{k-2}\frac{\partial\varphi_\sigma(y,x)}{\partial x_j}+$

$$+\frac{\partial r^k}{\partial x_j}\frac{\partial\varphi_\sigma(y,x)}{\partial x_j}+r^k\frac{\partial^2\varphi_\sigma(y,x)}{\partial x_j^2}.$$

There fore

$$\sum_{j=1}^{n}\frac{\partial^2 r^k\varphi_\sigma(y,x)}{\partial x_j^2}=\Delta r^k\varphi_\sigma(y,x)=knr^{k-2}\varphi_\sigma(y,x)+\sum_{j=1}^{n}k(x_j-y_j)(\frac{\partial r^{k-2}}{\partial x_j})\varphi_\sigma(y,x)$$

$$+2k\sum_{j=1}^{n}(x_j-y_j)r^{k-2}\frac{\partial\varphi_\sigma(y,x)}{\partial x_j}+\sum_{j=1}^{n}r^k\frac{\partial^2\varphi_\sigma(y,x)}{\partial\ x_j^2}$$

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as $\varphi_\sigma(y,x)$ harmonic function в R^m by variable x including point y

$$\Delta r^k \varphi_\sigma(y,x) = (kn + k(k-2))r^{k-2}\varphi_\sigma(y,x) + 2kr^{k-2}\sum_{j=1}^{n}(x_j - y_j)\frac{\partial\varphi_\sigma(y,x)}{\partial x_j}$$

If $\varphi_{\sigma,1}(y,x) = (kn + k(k-2))\varphi_\sigma(y,x) + 2k\sum_{j=1}^{n}(x_j - y_j)\frac{\partial\varphi_\sigma(y,x)}{\partial x_j}$

Marking , $c = (kn + k(k-2)) = k(n+k-2)$, we have

$$\frac{\partial\varphi_{\sigma,1}(y,x)}{\partial x_i} = c\frac{\partial}{\partial x_i}\varphi_\sigma(y,x) + 2k\left((x_1 - y_1)\frac{\partial^2\varphi_\sigma(y,x)}{\partial x_1\partial x_i} + \ldots + (x_{i-1} - y_{i-1})\frac{\partial^2\varphi_\sigma(y,x)}{\partial x_{i-1}\partial x_i}\right) +$$

$$+ 2k\left(\frac{\partial\varphi_\sigma(y,x)}{\partial x_i} + (x_i - y_i)\frac{\partial^2\varphi_\sigma(y,x)}{\partial x_i^2} + \ldots + (x_n - y_n)\frac{\partial^2\varphi_\sigma(y,x)}{\partial x_n\partial x_i}\right)$$

In addition, for a second-order partial derivative

$$\frac{\partial^2\varphi_{\sigma,1}(y,x)}{\partial x^2{}_i} = \frac{\partial}{\partial x_i}\left(c\frac{\partial}{\partial x_i}\varphi_\sigma(y,x) + 2k\frac{\partial}{\partial x_i}\sum_{j=1}^{n}(x_j - y_j)\frac{\partial\varphi_\sigma(y,x)}{\partial x_j}\right) =$$

$$= c\frac{\partial^2\varphi_\sigma(y,x)}{\partial x_i^2} + 2k\left((x_1 - y_1)\frac{\partial^3\varphi_\sigma(y,x)}{\partial x_1\partial x_i^2} + \ldots + (x_{i-1} - y_{i-1})\frac{\partial^3\varphi_\sigma(y,x)}{\partial x_{i-1}\partial x_i^2}\right) +$$

$$+ 2k\left(2\frac{\partial^2\varphi_\sigma(y,x)}{\partial x_i^2} + (x_i - y_i)\frac{\partial^3\varphi_\sigma(y,x)}{\partial x_i^3} + \ldots + (x_n - y_n)\frac{\partial^2\varphi_\sigma(y,x)}{\partial x_n\partial x_i^2}\right)$$

since the harmonic function in a variable including the point

$$\Delta\varphi_{\sigma,1}(y,x) = \sum_{j=1}^{n}\frac{\partial^2\varphi_{\sigma,1}(y,x)}{\partial x_j^2} = 2\sum_{j=1}^{n}\frac{\partial^2\varphi_\sigma(y,x)}{\partial x_j^2} + \sum_{j=1}^{n}(x_j - y_j)\frac{\partial\Delta\varphi_\sigma(y,x)}{\partial x_j}$$

this implies the assertion of the lemma.

Consequence 1. For function $\Phi_\sigma(y,x)$ fair estimate

$$|\Phi_\sigma(y,x)| \le Cr^{2n-2}\alpha^{-m}\exp(\sigma y_m - a\cos\rho_1\beta_2 ch\rho_1\alpha), \quad \alpha \ge 1$$

$$|\Phi_\sigma(y,x)| \le C\sigma^{n-2}\left(r^{2n-m} + \alpha^{-1}r^{2n-m+1} + \sum_{p=1}^{n-2}\alpha^{-2p}r^{2(p+2)}\right)\exp(\sigma y_m - a\cos\rho_1\beta_2 ch\rho_1\alpha), 0 < \alpha \le 1$$

Theorem 2. For function $\varphi_\sigma(y,x)$ there is an inequality

$$\left|\frac{\partial}{\partial n}\varphi_\sigma(y,x)\right| \le \begin{cases} C\sigma^{n-1}\alpha^{-m}\exp(\sigma y_m - a\cos\rho_1\beta_2 ch\rho_1\alpha), \alpha \ge 1 \\ C\sigma^{n-1}\left(\frac{|\cos\theta| + r}{r^{m-1}} + \sum_{p=1}^{n-2}\alpha^{-2p-1}r^{-2(n-p-1)-1}\right)\exp(\sigma y_m - a\cos\rho_1\beta_2 ch\rho_1\alpha), 0 < \alpha \le 1 \end{cases}$$

α_i - guide cosines of the normal vector. We denote by the space of polyharmonic functions defined in the order, having continuous partial derivatives of the order up to the end points of the boundary and satisfying the condition:

$$\sum_{k=0}^{n-1}\left(|\Delta^k u(y)| + |\ grad\Delta^{n-k-1}u(y)\ |\right) \le C\exp(\exp\rho_2(y')). \quad (3)$$

Theorem -3. Function, $\Phi_\sigma(y,x)$ fixed $x \in D$ function $\Phi_\sigma(y,x)$ satisfies

$$\sum_{k=0}^{n-1}\int_{\partial D\setminus S}\left[|\Delta^k\Phi_\sigma(y,x)| - \left|\frac{\partial\Delta^k\Phi_\sigma(y,x)}{\partial \bar{n}}\right|\right]ds_y \le C(x)\varepsilon(\sigma),$$

where the constant depends on x and is the external normal to, when $\sigma \to \infty$.

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We denote by the space of polyharmonic functions defined in D of order n, having continuous partial derivatives of order 2n-1 up to the end points of the boundary and satisfying the condition: $\sum_{k=0}^{n-1}\left(\left|\Delta^k u(y)\right| + \left| \ grad\Delta^{n-k-1}u(y) \ \right|\right) \leq C\exp(\exp\rho_2(y'))$.

Theorem - 4. Let for the function $u \in B_{\rho_2}(D)$ at any point the inequality

$$\sum_{k=0}^{n-1}\left|\Delta^k u(y)\right| + \left|\frac{\partial\Delta^{n-1-k}u(y)}{\partial\overline{n}}\right| \leq C \ \exp\left(a\cos\rho_3\left(y_1 - \frac{h}{2}\right)\exp\rho_3\left|y'\right|\right) \quad (6)$$

where $\rho_1 < \rho_2 < \rho_3 < \rho$. Then for any point $x_0 \in D$ equality holds

$$u(x_0) = \sum_{k=0}^{n-1}\int_{\partial D}\left[\Delta^k\Phi(y,x_0)\frac{\partial\Delta^{n-1-k}u(y)}{\partial\overline{n}} - \Delta^{n-1-k}u(y)\frac{\partial\Delta^k\Phi(y,x_0)}{\partial\overline{n}}\right]ds.$$

Note that for arbitrary $F_i(y), G_i(y)$ task $(1)-(2)$ insoluble.

Reference

1. Sobolev S.L. Introduction to the theory of cubature formulas. M.Nauka 1974.
2. Lavrentiev M.M. About some incorrect problems of mathematical physics.
3. Yarmuhamedov Sh., Juraeva N.Yu. The Cauchy problem for polyharmonic functions. Partial differential equations and related problems of analysis and computer science. Proceedings of the international scientific conference. Tashkent November 16-19, 2004 p.301-302.
4. Yarmuhamedov Sh. The Cauchy problem for the polyharmonic equation. Reports of the Russian Academy of Sciences 2003 vol. 388 p. 162-165.
5. Juraeva N. Yu. Growing polyharmonic functions and the Cauchy problem. Moscow. International conference dedicated to the 60th anniversary of Yu. P. Soloviev. February 14-19, 2005 p. 74-77.
6. Juraeva N.Yu. On a Cauchy problem for growing polyharmonic functions. New directions in the theory of dynamical systems and ill-posed problems. International Conference. Samarkand October 19-20, 2007 p. 139-141.
7. Juraeva N.Yu. Integral representation for polyharmonic functions. Differential equation and their proposal. Materials of the republican scientific conference dedicated to the 100th anniversary of academician I.S. Kukles. Samarkand. 2007 p. 54-55.
8. Juraeva N.Yu. Ashurova. Z.R. Properties of polyharmonic functions. Samarkand. vol. 3, 2008 p. 8-11.
9. Juraeva N.Yu. On the integral representation of polyharmonic functions. Tashkent. ACADEMY OF SCIENCES OF THE REPUBLIC OF UZBEKISTAN No. 3, 2008 p. 18-20.
10. Juraeva N.Yu. The Cauchy problem for growing polyharmonic functions. International Conference "Inverse and Ill-posed Problems of Mathematical Physics" dedicated to the 75th anniversary of Academician M. M. Lavrentiev. Novosibirsk 2007.p. 65-67.
11. Juraeva N.Yu. Growing polyharmonic functions and the Cauchy problem. Uzbek mathematical journal. No. 2, 2009, p. 70-74.