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Mathematical Modeling and Investigation of Optimal Operating Modes of a Micro Hydropower Plant Hydraulic Turbine

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Abstract
This paper investigates the determination of optimal operating parameters of a micro-hydropower plant hydraulic turbine equipped with bucket-type blades, designed to operate under variable water discharge and head conditions. The process of converting the hydraulic and kinetic energy of the water flow into mechanical energy is mathematically modeled, and the interrelationships among the main geometric and energy parameters of the hydraulic turbine are analyzed. The relationships between water discharge, hydraulic head, turbine rotational speed, torque, mechanical power, and efficiency are determined based on analytical expressions. In addition, a computational algorithm is developed to determine the optimal installation angle of the bucket-type blades and to select the optimal operating conditions of the hydraulic turbine. Based on the computational results, the optimal turbine parameters are evaluated for operating conditions with a water discharge varying within the range of 0.003–0.009 m^3/s. It is determined that, at a hydraulic head of 30 m and a total water discharge of 0.018 m^3/s, the optimal rotational speed of each hydraulic turbine is 540 rpm, while the mechanical power is 2150 W. The total mechanical power transmitted from the two hydraulic turbines to the generator is 4300 W. These results demonstrate the feasibility and effectiveness of the proposed turbine configuration for operation under variable water-flow conditions.
Keywords: micro hydropower plant, bucket-type blade, hydraulic turbine, mathematical modeling, water discharge, hydraulic head, rotational speed, mechanical power, torque, efficiency, optimization.

1. Introduction

Efficient utilization of renewable energy sources, particularly the conversion of the hydraulic energy of small water flows into electrical energy, is one of the important areas of modern energy development. Small and micro-hydropower plants provide opportunities to utilize available water resources, supply electricity to remote areas, and make effective use of local energy sources. The energy efficiency of such systems depends on the compatibility and interaction of water discharge, effective hydraulic head, flow velocity, and the design parameters of the hydraulic turbine.

The use of mathematical modeling methods in the design of hydraulic turbines makes it possible to theoretically evaluate various operating conditions and optimize the design parameters [1, 2]. In this approach, hydraulic power can be determined from the water discharge and hydraulic head, while the turbine torque and mechanical power can be calculated based on the tangential velocity components of the flow [3]. This approach provides a theoretical basis for the optimal selection of turbine rotational speed and blade installation angle [2, 3]. For the micro-hydropower plant under investigation, the relationships among water discharge, hydraulic head, rotational speed, torque, mechanical power, and efficiency were also determined using a mathematical model.

Based on the above considerations, the aim of this study is to determine the optimal energy parameters of a micro-hydropower plant hydraulic turbine equipped with bucket-type blades under variable water discharge and hydraulic head conditions using mathematical modeling. The study focuses on analyzing the dependence of turbine rotational speed, mechanical power, torque, and efficiency on water discharge and hydraulic head, with particular emphasis on identifying efficient operating conditions.

2. Material and methods

The efficiency of electricity generation in small hydropower plants depends on the degree to which the energy of the water flow is utilized. In hydraulic turbines equipped with bucket-type blades, the kinetic energy of the water is directed onto the blades and converted into rotational mechanical energy. In this process, the flow

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velocity and direction, as well as the geometric characteristics of the blades, are the key factors determining turbine efficiency. Mathematical and computational modeling methods are used to evaluate the turbine operating process. While the mathematical model enables the determination of flow velocity, impact angle, power, torque, and efficiency, computational modeling is used to analyze flow dynamics and the interaction between the water flow and the buckets. These methods make it possible to optimize turbine parameters and improve overall energy efficiency.

Hydraulic turbines equipped with bucket-type blades are effectively applied in small hydropower plants, particularly under conditions of limited water discharge and relatively high hydraulic head. In this study, turbine power, water discharge, flow velocity, blade geometry, and efficiency were considered as the main performance indicators.

The hydraulic turbine design is characterized by a number of geometric parameters. The main parameters, illustrated using a single bucket as an example, are presented in **Figure 1**.

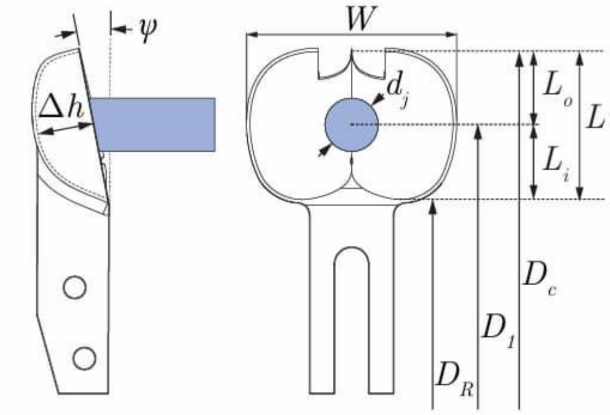

Figure 1. Geometric Parameters of the Bucket-Type Blade

Figure 1 presents the main geometric parameters of the bucket-type blade. These parameters include the following:

D_C — the outer diameter of the bucket-type blade, i.e., the diameter between the outermost points of the hydraulic surface;

D_1 — the turbine diameter corresponding to the zone where the water flow exiting the nozzle interacts with the bucket-type blade;

D_R — the inner diameter corresponding to the innermost point of the hydraulic surface and located close to the supporting section of the blade;

d_j — the nozzle outlet diameter, i.e., the size of the cross-section through which the water flow exits;

W — the overall width of the bucket-type blade;

α — the installation angle of the bucket-type blade relative to the water flow exiting the nozzle, which determines the distribution of the flow in the peripheral direction;

Δh — the depth of the bucket-type blade;

L — the total length of the hydraulic surface of the bucket-type blade in the radial direction (1).

$$L = \frac{D_c - D_R}{2} \tag{1}$$

L_i — the length of the hydraulic surface of the bucket-type blade located within the inner region of the circular diameter (D_1) of the water flow at the nozzle outlet (2).

$$L_i = \frac{D_1 - D_R}{2} \tag{2}$$

L_o — the length of the hydraulic surface of the bucket-type blade located within the outer region of the circular diameter (D_1) of the water flow at the nozzle outlet (3).

$$L_0 = \frac{D_c - D_1}{2} \tag{3}$$

The optimal angle of the hydraulic turbine was determined through multi-stage iterative calculations. The computational methodology proposed by Zhang [4, pp. 589–596] was adopted as the basis for the study. Since the mathematical foundations and complete solution procedure of this method have been described in detail in the cited reference, the present study focuses primarily on the key computational results and their analysis. According to the operating principle of the Pelton turbine, in order to utilize the energy of the water flow to the maximum extent, the turbine rotational speed should correspond to approximately 50% of the inlet flow velocity (4) [5, pp. 1–324], i.e.:

$$u = \frac{v_1}{2} \tag{4}$$

$$v_1 = c_v\sqrt{2gH_0} \tag{5}$$

where: u — tangential (rotational) velocity, m/s; V_1 — water flow velocity at the nozzle outlet, m/s; g — acceleration due to gravity, m/s²; H — effective hydraulic head from the headwater to the nozzle, m.

Since the water flow travels only a short distance along the surface of the bucket-type blade, the relative velocity W during the interaction between the flow and the blade was assumed to remain constant.

$$u_1 = \omega\frac{D_1}{2} = \omega R_1 = \frac{v_1}{2} \tag{6}$$

$$w_1 = v_1 - u_1 = \frac{v_1}{2} \tag{7}$$

where ω — angular velocity, rad/s.

The diameter of the water jet (reactive flow) exiting the nozzle is determined using the following expression:

$$d_j = \frac{Q}{Z\sqrt{2gH_e}} \tag{8}$$

where Z — the number of nozzles (water jets at the nozzle outlets); Q — water discharge, m³/s.

An x–y coordinate system was employed to analyze the hydrodynamic processes in the bucket-type turbine. Using this coordinate system, the tangential and radial velocity components resulting from the interaction between the water jet exiting the nozzle and the blade were determined. This approach enabled the evaluation of the distribution of impulse and torque, as well as the theoretical analysis of velocity triangles, turbine power, and hydraulic efficiency (Figure 2). The initial point of interaction between the water jet exiting the nozzle and the bucket-type blades, i.e., the primary point of water entry, occurred at an angle αA.

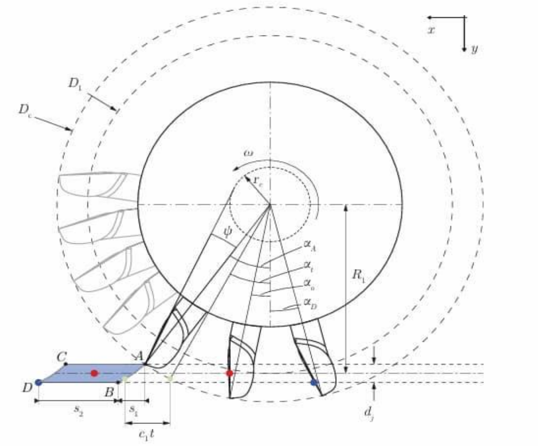

Figure 2. Geometric Scheme of the Interaction between the Bucket-Type Blades and the Water Jet Exiting the Nozzle

In this case, the position of the bucket-type blade was defined by point A. The projections and lengths of the water-jet segment exiting the nozzle onto the x-axis were determined using the following equations.

$$\cos(\alpha_A) = \frac{R_1 - \frac{d_j}{2}}{R_c} \tag{9}$$

$$S_1 = \frac{d_j(\frac{1}{u_1} - 1)}{\sqrt{\left(\frac{R_c}{R_1}\right)^2 - 1}} \tag{10}$$

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$$S_2 = \frac{R_1}{u_1}\frac{2\pi}{N} \tag{11}$$

where: α_A — initial interaction angle between the water flow and the bucket-type blade, degrees (°); R_1 — turbine radius, m; R_c — outer radius of the turbine, m; S_1 — length of the water-flow segment from its initial interaction with the blade until separation from the blade, m; S_2 — length of the water-flow segment until its interaction with the next blade, m; N — number of bucket-type blades.

Equation (12) describes the relationship between the motion of a water particle along the flow exiting the nozzle and the angular position of the turbine, and is used to evaluate the interaction between the water particle and the turbine. According to Figure 2, point A is taken as the origin of the coordinate system, while the values of x and y define the current position of the water particle within the flow.

$$\frac{x}{R_1} = \frac{\alpha_A - \alpha_t}{u_1} - (1 - \frac{d_j}{2R_1})\tan(\alpha_A) + [1 - \frac{d_j}{2R_1} + \frac{y}{R_1} + \frac{r_e}{R_1 \sin(\alpha_t - \psi)}]\tan(\alpha_t - \psi) \tag{12}$$

where: $\frac{x}{R_1}$ — horizontal position of the nozzle outlet flow or interaction point relative to the radius, m; α_t — angular coordinate of the subsequent interaction point, degrees (°); $\tan\alpha_A$ — inclination angle of the water-flow inlet direction relative to the coordinate axis; $\frac{y}{R_1}$ — vertical position of the flow or interaction point relative to the radius, m; $\sin(\alpha_t - \psi)$ — trigonometric parameter representing the angular difference between the flow direction and the blade deflection direction;

ψ — deflection angle of the bucket-type blade, degrees (°); r_e — horizontal distance between the turbine axis of rotation and the blade tip, m; i.e.:

$$r_e = R_c \tan\psi. \tag{13}$$

If α_0 has a small value, Equation (15) can be expressed in terms of Equations (13) and (14).

$$\alpha_t = \frac{\frac{x}{R_1} - \frac{\alpha_A}{u_1} - \frac{R_c \sin\psi}{R_1} + (1 - \frac{d_j}{2R_1})\tan(\alpha_A) + (1 - \frac{d_j}{2R_1} + \frac{y}{R_1})\psi}{1 - \frac{1}{u_1} - \frac{d_j}{2R_1} + \frac{y}{R_1}} \tag{14}$$

The angular position α_0 of the bucket-type blade at the moment of interaction with the center of the water-flow volume exiting the nozzle was determined using Equation (15). The corresponding geometric positions were assumed as follows:

$$x = \frac{S_1+S_2}{2} \quad \text{va} \quad y = \frac{d_j}{2}$$

$$\alpha_0 = \frac{\frac{S_1+S_2}{2R_1} - \frac{\alpha_A}{u_1} - \frac{R_c \sin\psi}{R_1} + \left(1 - \frac{d_j}{2R_1}\right)\tan\alpha_A + \psi}{(1 - \frac{1}{u_1})} \tag{15}$$

To evaluate the energy potential of the water flow entering the turbine, the total hydraulic power, which depends on the water discharge and hydraulic head, was adopted as the primary calculation parameter.

$$P_j = \rho g Q H \tag{16}$$

where: P_j — hydraulic power of the water flow, kW; ρ — fluid density, kg/m³; H — effective hydraulic head, m.

The mechanical power of the turbine was determined based on Equation (17) from the difference between the tangential velocity components at the inlet and outlet sections of the bucket-type blade. The change in the momentum of the water particles was taken into account to determine the mechanical power generated by the turbine. The value of α_0 was determined using Equation (15).

$$P_t = \rho Q u(V_1 \cos\alpha_0 - u)(1 + \cos\beta) \tag{17}$$

The efficiency of the bucket-type blade micro-hydropower plant was determined using Equation (21). It represents the ratio of the mechanical power generated by the hydraulic turbine runner to the hydraulic power of the water flow and is based on the change in the tangential velocity components of the flow.

$$P_t = M\omega = \dot{m}u\left(V_{u_1} - V_{u_2}\right) \tag{18}$$

$$M = \frac{P_t}{\omega}; \omega = \frac{u}{r} \tag{19}$$

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$$\dot{m} = \rho Q \tag{20}$$

$$\eta = \frac{P_t}{P_j} = \frac{M\omega}{\rho g Q H} = \frac{\dot{m}u(V_{u_1} - V_{u_2})}{\rho g Q H} = \frac{u(V_1 \cos\alpha_0 - u)(1 + \cos\beta)}{gH} \tag{21}$$

where: P_t — mechanical power, kW; M — torque generated on the hydraulic turbine shaft, N·m; ω — angular velocity, rad/s; $\dot{m}$ — mass flow rate, kg/s; V_{u1} — tangential velocity component at the inlet, m/s; V_{u2} — tangential velocity component at the outlet, m/s; η — hydraulic turbine efficiency; $V_1 \cos\alpha_0$ — tangential component of the water-flow inlet velocity at the nozzle outlet; β — flow deflection angle at the bucket-type blade, degrees (°).

Based on the mathematical model presented above, the conversion of the kinetic energy of the water flow into mechanical energy, as well as the effects of turbine rotational speed and inertia, were theoretically evaluated for the Pelton turbine. The proposed model enables the optimization of the main parameters of bucket-type hydraulic turbines for micro- and small-scale hydropower plants, including the selection of water-flow velocity, the number of buckets, and the nozzle outlet angle.

3. Result and Discussion

In Figure 3, the dependence of turbine power and efficiency on the bucket-type blade installation angle was modeled based on the following initial conditions: hydraulic head in the range of 10 m ≤ H ≤ 30 m and water discharge in the range of 0.005 m³/s ≤ Q ≤ 0.009 m³/s. At a hydraulic head of H = 10 m and a water discharge of Q = 0.005 m³/s, the maximum mechanical power was obtained at ψ = −30°, corresponding to P = 940 W (Figure 3a). At H = 20 m and Q = 0.007 m³/s, the maximum mechanical power was also obtained at ψ = −30°, with P = 2660 W (Figure 3b). At H = 30 m and Q = 0.009 m³/s, the maximum mechanical power reached P = 4880 W at ψ = −30° (Figure 3c). The bucket-type blade installation angle of ψ = −30° was found to provide a hydraulic turbine efficiency of approximately η = 92% (Figure 3d).

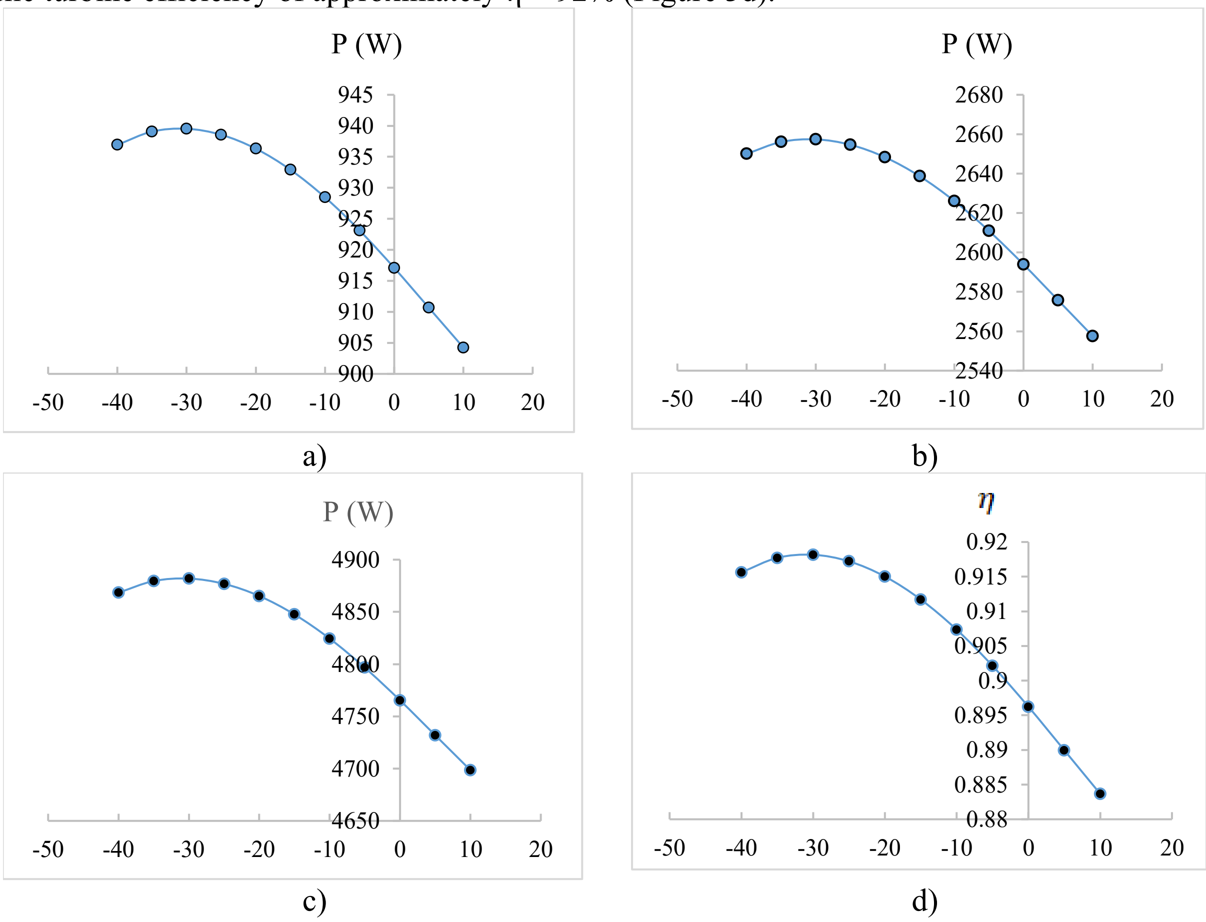
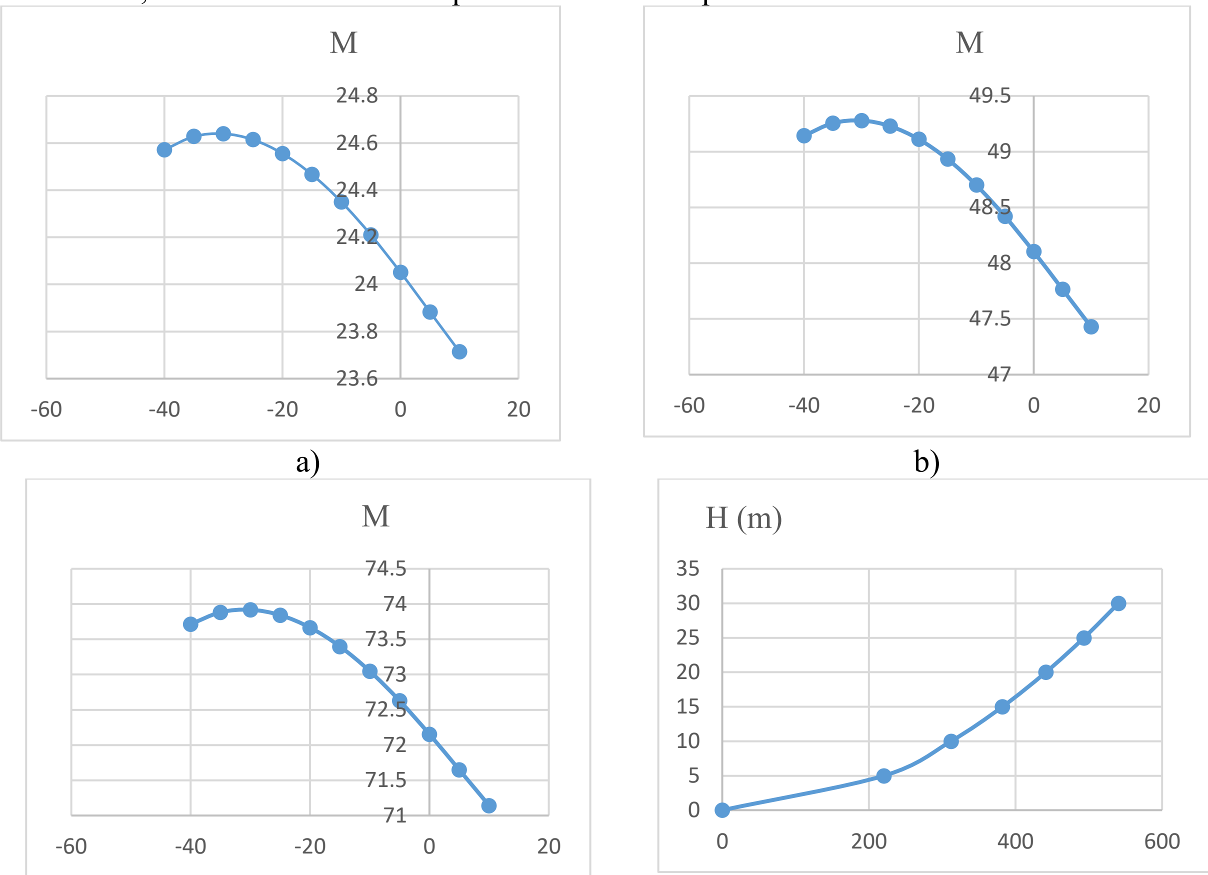


Figure 3. Variation of Turbine Power and Efficiency with the Installation Angle of the Bucket-Type Blades

Figure 4 presents the dependence of the hydraulic head H on the hydraulic turbine torque M and the number of turbine runner revolutions N. The analysis was performed under the initial conditions of a hydraulic head in the range of 10 m ≤ H ≤ 30 m and a water discharge in the range of 0.005 m³/s ≤ Q ≤ 0.009 m³/s. In Figure 4a, at H = 10 m, Q = 0.005 m³/s, and ψ = −30°, the maximum torque reached M = 25 N·m. In Figure 4b, at H = 20 m, Q = 0.007 m³/s, and ψ = −30°, the maximum torque reached M = 49 N·m. In Figure 4c, at H = 30 m, Q = 0.009 m³/s, and ψ = −30°, the maximum torque reached M = 74 N·m. Figure 4d shows that at a hydraulic head of H = 30 m, the maximum turbine speed was N = 540 rpm.



a) b)

c) d)

Figure 4. Effect of Hydraulic Head on Turbine Torque and Runner Rotational Speed as a Function of the Installation Angle of the Bucket-Type Blades

Based on the modeling results presented above, an algorithm was developed to determine the effects of water discharge and turbine rotational speed on hydraulic head, torque, mechanical power, efficiency, and the optimal installation angle of the blades (Figure 5). The penalty function method was employed to solve the optimization problem.

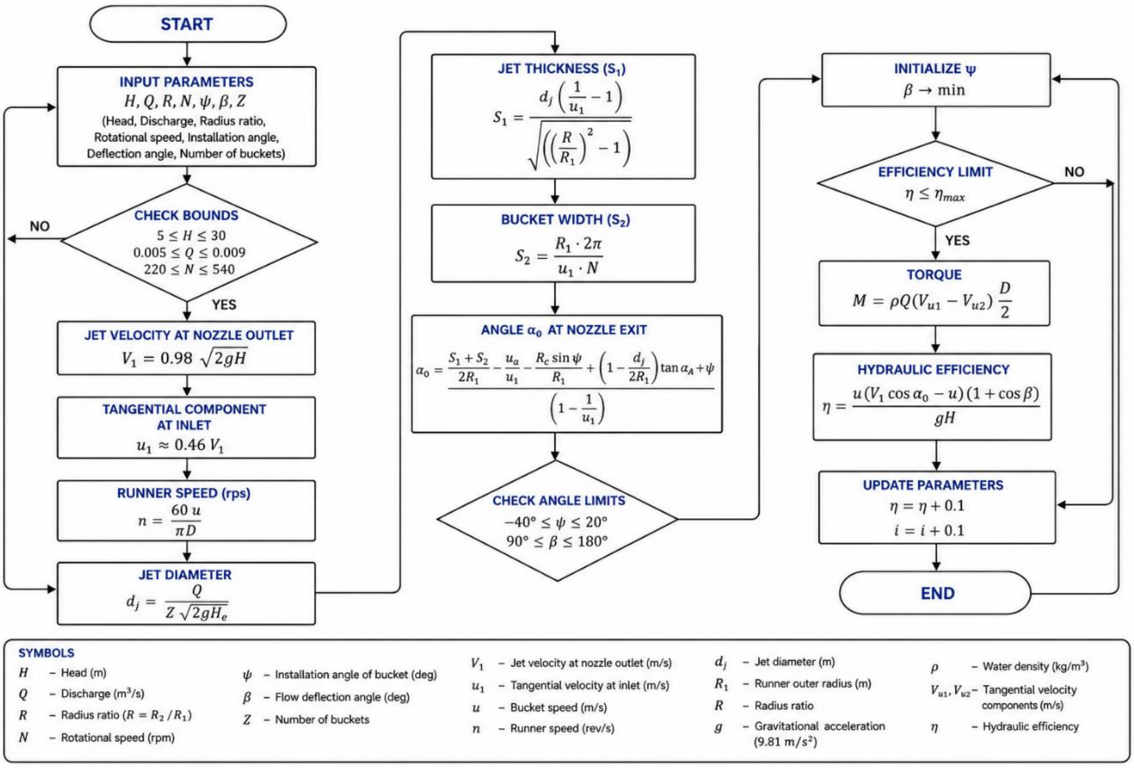


Figure 5. Algorithm for Determining the Dependence of Water Discharge and Turbine Rotational Speed on the Parameters (H, Q, n, M, P, η, ψ_{opt}).

Based on this algorithm:

1.The optimal values of the hydraulic head H, **t**orque M, mechanical power P, and efficiency $\eta$of the bucket-type hydraulic turbine can be determined as a function of the turbine rotational speed $n$and water discharge Q.

2.The optimal blade installation angle ψ_{opt}can be determined as a function of the turbine hydraulic head H_t, torque M, mechanical power P, rotational speed n, and water discharge Q.

Based on the calculations and analyses performed, the relationships between the hydraulic head, torque, mechanical power, efficiency, and optimal blade installation angle of the bucket-type hydraulic turbine and the water discharge and turbine rotational speed were determined. In this case, the calculations were performed using a turbine diameter of 0.42 m, a runner disc diameter of 0.36 m, and 25 blades, with the water discharge varying within the range of 0.003–0.009 m³/s, based on the geometric relationships and mathematical expressions presented above. Thus, it was determined that at the maximum water discharge of 0.009 m³/s, the optimal geometric blade installation angle is $\psi_{opt} = -30°$.

Table 1 presents the optimal operating parameters of the proposed bucket-type blade micro-hydropower plant corresponding to different water discharge and hydraulic head values.

Table 1

№	Q_1, m^3/s	n_1, rpm	P_1, W	H, m	ψ_{opt}, deg	Q_2, m^3/s	n_2, rpm	P_2, W
1	0,009	540	2150	30	-30	0,009	540	2150

Based on the results of the conducted studies, it was determined that, for the proposed bucket-type blade micro-hydropower plant, at a water discharge of 0.018 m³/s and a hydraulic head of 30 m, the optimal rotational speed of each hydraulic turbine is 540 rpm, while the mechanical power is 2150 W. Consequently, the total mechanical power transmitted to the generator is 4300 W. Using a belt-drive transmission with a 2:1 speed ratio, the rotational speed of the generator is increased to 1080 rpm.

Therefore, in the proposed micro-hydropower plant with a Pelton-type hydraulic turbine, the turbine rotational speed, hydraulic head, torque, mechanical power, efficiency, and optimal blade installation angle were

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determined as functions of the variable water discharge and low-head operating conditions, thereby ensuring efficient and stable operation of the system.

4. Conclusion

The conducted study developed a mathematical model of the operating processes of a bucket-type blade micro-hydropower turbine operating under variable water discharge and hydraulic head conditions. The relationships among water discharge, hydraulic head, rotational speed, torque, mechanical power, and efficiency were determined, enabling the evaluation of the turbine's operating modes.

Based on the computational results, an algorithm was developed to determine the optimal blade installation angle and turbine rotational speed. The penalty function method was employed for optimization, and the effects of water discharge and rotational speed on the hydraulic turbine's energy performance were evaluated. It was determined that, at a hydraulic head of 30 m and a total water discharge of 0.018 m^3/s, the optimal rotational speed of each hydraulic turbine is 540 rpm, with a mechanical power of 2150 W. The total mechanical power transmitted from the two hydraulic turbines to the generator is 4300 W. Using a belt-drive transmission, the rotational speed is doubled, increasing the generator rotational speed to 1080 rpm.

Thus, the developed mathematical model and optimization algorithm provide a practical basis for selecting the optimal operating parameters of the proposed bucket-type blade micro-hydropower plant and ensuring its efficient and stable operation under variable water discharge and low-head conditions.

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