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To approximate solution to a problem of filtration theory for small values of TIME

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Abstract: The problem of unsteady filtration, which subsequently modeled as a boundary value problem with an unknown moving boundary section. We prove the existence and uniqueness of the solution of the problem in the small.

Key words and phrases: filtration, evaporation, interface, steady-state approximation, the Frechet derivative, integral equation of Voltaire, the existence and uniqueness of solutions.

1. Statement of the problem.

An increase in the water horizon in hydraulic structures causes a rise (backwater) of the surface of the soil flow in the adjacent territories and in some cases creates a threat of flooding of cities, settlements, as well as salinization and waterlogging, valuable lands for agriculture. At the same time, the arrangement of reservoirs and reservoirs leads to rational redistribution for electricity, irrigation and watering of lands, water supply, improves climatic conditions, etc.

When filling new canals and reservoirs, there is a rise, and during their operation, both the rise and fall of the level in them can occur. If the initial level in the reservoir is located below the critical level and there is a rise in the level in the channel, then after a while in the canals (reservoir), the level will reach a critical value. In the future, there are two areas of motion, with a movable interface in one of which there is evaporation, and in the other it is absent. h_l $h_{кр}$ $t_0 > 0$ $x = l(t)$

As is known [1], the level within the hydraulic theory satisfies the Boussinesq equation $h(x,t)$

$$\mu \frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(K \cdot h \frac{\partial h}{\partial x} \right) + W \quad , \qquad (1)$$

where is the coefficient of water loss (effective porosity), is the filtration coefficient of the formation, is an expression that takes into account the vertical velocity at the lower base of the formation, infiltration and evaporation from the free surface. μ K W

Consider the problem of filtration near new canals and reservoirs in the presence of evaporation from the groundwater mirror, depending on the depth of the groundwater level (UGV) and time according to the law ε

$$\varepsilon = \begin{cases} f\left(h - h_{кр}, t\right), \ h > h_{кр}; \\ \qquad\qquad 0, \ h \le h_{кр}. \end{cases} \qquad (2)$$

where is a function that is smooth enough for the first argument and piecewise smooth for the second argument, which has the following properties: $f(z,t)$

$$f(0,t) = 0, \quad f(z,t) > 0, \quad f_z'(z,t) > 0, \quad z > 0 \quad . \qquad (3)$$

To simplify the study in equation (1), we carry out separate linearization:

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$$a^2(x)=\begin{cases} a_1^2 = K\cdot \bar{h}_1/\mu,\ h>h_{кр}\ \left(0<x<l(t)\right), \\ a_2^2 = K\cdot \bar{h}_2/\mu, h>h_{кр}\ \left(l(t)<x<\infty\right); \end{cases} \qquad (4)$$

where and are some average values, $\bar{h}_1\ \bar{h}_2\ h(x,t)$ respectively, from the intervals and ; ; – level values in the channel. $[h_{кр},h_m]\ [h_l,h_{кр}]\ h_m=\max_t \psi(t)\ \psi(t)$

Taking into account the assumptions made above, the problem of determining the unknown boundary of the section is formulated as follows. $h(x,t)\ x=l(t)$

Find the functions and , under the following conditions $h(x,t)\ l(t)\ l(t_0)=0$

$$\frac{\partial h}{\partial t}=a^2(x)\frac{\partial^2 h}{\partial x^2}-\frac{\varepsilon}{\mu},\ (x,t)\in\Omega_{t_0}^{\infty}\setminus\{x:x=l(t)\} \quad , \qquad (5)$$

$$h(0,t)=\psi(t);\quad h(x,t_0)=\varphi(x);\quad h\big|_{x\to+\infty}=h_l \quad , \qquad (6)$$

$$h\left(l(t)-0,t\right)=h\left(l(t)+0,t\right)=h_{кр}=const>0 \quad , \qquad (7)$$

$$\bar{h}_1\frac{\partial h(x,t)}{\partial x}\bigg|_{x=l(t)-0}=\bar{h}_2\frac{\partial h(x,t)}{\partial x}\bigg|_{x=l(t)+0} \quad , \qquad (8)$$

where the functions are smooth enough, and ; $\varphi,\ \psi\ \psi(t_0)=\varphi(0)=h_{кр}$ $\Omega_{t_0}^{\infty}=\{(x,t):\ 0<x<\infty,\ t_0<t<T\}$ There is a moment when the water level in the canal reaches the mark and is determined by formulas (4) and (2), respectively. $t_0\ h_{кр}=h(0,\ t_0),\ a^2(x)\ \varepsilon$

Problem (5) – (8) from problems of the Stefan type [2] is that at the known interface the flow rate (flow) is continuous (condition (8)), while in the problems of the Stefan type the flow rate is discontinuous and this gap is proportional to the speed of advance of the movable boundary (front).

The requirement and the specified difference from the condition of the Stefan type, i.e. the conditions at the unknown boundary, the resolved relative to the desired curve or its derivative create additional difficulties in the study of the problem (5) - (8). In this regard, it is also of independent mathematical interest. $l(t_0)=0$

2. Solution of problem (5) – (8) by the method of quasi-stationary approximation.

The essence of the method lies in the fact that the movable boundary is "frozen", i.e. the usual problem of conjugation with the vertical interface is relied on and solved. Then, by substituting the function together and using the condition on the boundary, an equation is obtained to determine . $l(t)=l(s)=const\ \ x=l(s)$ $l(s)\ l(t)\ x=l(t)\ l(t)$

And so, believing, consider the following two-layer problem on a semistraight line, the solution of which depends on : $l(t)=l(s)\neq 0\ \ x>0\ l(s)$

$$\frac{\partial u^{(s)}}{\partial t}=a^2(x)\frac{\partial^2 u^{(s)}}{\partial x^2}-\frac{\varepsilon}{\mu},\ (x,t)\in\Omega_{t_0}^{\infty}\setminus\{x:x=l(s)\}, \qquad (9)$$

$$u^{(s)}\big|_{x=0}=\psi(t);\quad u^{(s)}\big|_{t=t_0}=\varphi(x);\quad u^{(s)}\big|_{x\to+\infty}=0 \ ,$$

(10) ,

$$(11)\ [u^{(s)}]\big|_{x=l(t)}=0,\ \bar{h}_1\frac{\partial u^{(s)}}{\partial x}\bigg|_{x=l(s)-0}=\bar{h}_2\frac{\partial u^{(s)}}{\partial x}\bigg|_{x=l(s)+0}$$

where and is determined by formulas (2), (4) taking into account $\varepsilon\ a(x)\ l(t)=l(s)$; have the properties: , , are continuous functions in their domain and in addition , , , , , , ; . $\varphi,\ \psi,\ f\ \ f_{zz}''(z,t)\ \varphi''(x)\ \psi''(t)\ \varphi(x)$

$\psi(t)$ $\psi'(t)$ $\varphi''(x) > 0$ $\varphi'(x) < 0$ $\psi(t_0) = \varphi(0) = \psi_0 = const$ $u(x,t) = h(x,t) - h_l$

The function that determines the position of the UGV at the moment is found as a solution to the first boundary value problem for the equation of thermal conductivity on a semistraight line [3]. $\varphi(x)$ t_0 $x > 0$

$$\frac{\partial u}{\partial t} = a_2^{\,2} \frac{\partial^2 u}{\partial x^2}\ , , , 0 < x < \infty\ 0 < t \le t_0$$

$$u\big|_{t=0} = 0\ , ,\ u\big|_{x=0} = \psi(t)\ u\big|_{x\to+\infty} = 0$$

and has the form

$$u(x,t_0) = \varphi(x) = \frac{x}{2a_2\sqrt{\pi}} \int_0^{t_0} \frac{\psi(\tau)\exp\{-x^2/4a_2^{\,2}(t-\tau)\}}{(t_0-\tau)^{3/2}} d\tau .$$

In [2], using the Laplace–Carson integral transformation, the Green's function of the two-layer first boundary value problem on a semistraight line for the thermal conductivity equation with a piecewise constant coefficient in the case of . In the case of , it looks like this: $x > 0$ $l(s) = 1$ $l(s) \ne 1$ $l(s) > 0$

$$g(x,t,\xi,\tau;l(s)) = \begin{cases} g_1(x,t,\xi,\tau;l(s)), & 0 < x,\xi < l(s), \\ g_2(x,t,\xi,\tau;l(s)), & 0 < \xi < l(s) < x < \infty, \\ g_3(x,t,\xi,\tau;l(s)), & 0 < x < l(s) < \xi < \infty, \\ g_4(x,t,\xi,\tau;l(s)), & l(s) < x,\xi < \infty. \end{cases} \qquad (12)$$

Functions, are represented in the form of series, due to the cumbersomeness of expressions for which they are not given here. $g_i(x,t,\xi,\tau;l(s))$ $i = 1,2,3,4$

Using the integral representation [4] of the solution of problem (9)–(11), it is not difficult to obtain the following integral equation

$u^{(s)}(x,t) = A^{(s)}u^{(s)} + B(x,t;l(s))$.

In here

$$A^{(s)}u^{(s)} = \frac{1}{\mu}\int_{t_0}^{t} d\tau \int_0^{l(s)} g(x,t,\xi,\tau;l(s)) \cdot f(u^{(s)} - \psi_0, \tau) d\xi\ ,$$

$$B(x,t;l(s)) = \int_0^{+\infty} g(x,t,\xi,t_0;l(s)) \cdot \varphi(\xi) d\xi + a_1^{\,2} \int_{t_0}^{t} \left.\frac{\partial g(x,t,\xi,\tau;l(s))}{\partial \xi}\right|_{\xi=0} \cdot \psi(\tau) d\tau .$$

We take as an approximate solution of the problem (5)–(8) the functions and the definable solution of the system of functional equations $u(x,t)$ $l(t)$

$$\begin{cases} u(x,t) = Au + B(x,t;l(t)), \\ \psi_0 = \left[Au + B(x,t;l(t))\right]\big|_{x=l(t)}. \end{cases} \qquad (13)$$

In here

$$Au + B(x,t;l(t)) \approx \left(A^{(s)}u^{(s)} + B(x,t;l(s))\right)\Big|_{l(s)=l(t)} .$$

The results of the numerical solution of problems differ from the exact solution obtained in the quasistationary approximation for sufficiently small values of time [2]. However, the question of estimating the error of the method remains open.

System (13) is written in the form of a matrix operator equation:

$$P\begin{pmatrix} u \\ l \end{pmatrix} = \begin{pmatrix} P_1(u,l) \\ P_2(u,l) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\ , \qquad (14)$$

Where is

$P_1(u,l) = u - Au - B(x,t;l(t))$; $P_2(u,l) = \psi_0 - \left[Au + B(x,t;l(t))\right]\Big|_{x=l(t)}$.

We assume that the scopes and values of the operator are, respectively, spaces and , where $P\begin{pmatrix}u\\l\end{pmatrix}$ C^*

C

$$C^* = C_1 \oplus C_2, \ , \ ; \ . C_1 = \left\{u(x,t) : u \in C_{[0,b]\times[t_0,T]}\right\}$$

$$C_2 = \left\{l(t) : l(t) = m(t)(t-t_0)^{\frac{1}{2}+\alpha}, \ m(t) \in C_{[t_0,T]}, \ \frac{1}{2} \le \alpha < 1\right\} b = \max_{t_0 \le t \le T} l(t) = l(T)$$

A sign is understood as a direct product. $\oplus$

Let's define the norms in spaces and: C^*, C_1 C_2

$$\|u\|_{C_1} = \max_{x,t} |u(x,t)|; \ ; \|l(t)\|_{C_2} = \max_t \left|(t-t_0)^{-1/2} \cdot l(t)\right|$$

$$\left\|\begin{pmatrix}u\\l\end{pmatrix}\right\|_{C^*} = \|u\|_{C_1} + \|l(t)\|_{C_2}; \ . \left\|P\begin{pmatrix}u\\l\end{pmatrix}\right\|_C = \|P_1(u,l)\|_C + \|P_2(u,l)\|_C$$

It can be shown that the nonlinear operation, $P\begin{pmatrix}u\\l\end{pmatrix}$ with the assumptions made above regarding the functions , , and values sufficiently close to , has a single zero, i.e. exists in such an element that . φ ψ f t

$$t_0 \ C^* \begin{pmatrix}u^*\\l^*\end{pmatrix} P\begin{pmatrix}u^*\\l^*\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

Let us apply the Newton-Kantorovich method to equation (14), i.e. we construct the Newtonian process for functional equations.

First, let's look at the case of . $0 < x < l(t)$

Then the $l_0(t) = 2a_1(t-t_0)^{\alpha+1/2}$ $\frac{1}{2} \le \alpha < 1$ $u_0(x,t) = B(x,t;l_0(t))$ first approximation of Newton's process is found by the solution of the following matrix equation with respect to corrections

$\Delta u(x,t) = u_1(x,t) - u_0(x,t)$ and: $\Delta l = l_1(t) - l_0(t)$

$$\begin{pmatrix} P'_{1u}(u_0,l_0)\cdot\Delta u + P'_{1l}(u_0,l_0)\cdot\Delta l \\ P'_{2u}(u_0,l_0)\cdot\Delta u + P'_{2l}(u_0,l_0)\cdot\Delta l \end{pmatrix} = \begin{pmatrix} -P_1(u_0,l_0) \\ -P_2(u_0,l_0) \end{pmatrix} . \qquad (15)$$

Here, –Fréchet derivatives [5]. $P'_{iu}(u_0,l_0)$ $P'_{il}(u_0,l_0)$ $(i=1,2)$

We evaluate the norms , $\left\|P\begin{pmatrix}u_0\\l_0\end{pmatrix}\right\|$ and , which are involved in the inequality that ensures the convergence of the Newtonian process. $\left\|\left(P'\begin{pmatrix}u_0\\l_0\end{pmatrix}\right)^{-1}\right\| \left\|P''\begin{pmatrix}u\\l\end{pmatrix}\right\|$

By virtue of the assumptions about the smoothness , and , after some transformations of the terms of the left side (12) we obtain φ ψ f

$$\left\|P\begin{pmatrix}u_0\\l_0\end{pmatrix}\right\| \le \eta_1 \equiv M_P \cdot (t-t_0)^{\alpha} \qquad , \quad , \qquad (16)\ \frac{1}{2} \le \alpha < 1$$

where is a positive constant that depends on the data of the problem. M_P

The solution of the matrix equation (14) is equivalent to a system of linear integral equations of the second kind. According to the general theory of Voltaire's system of integral equations of the second kind, it is solvable, i.e. there is an inequality

$$\left\|\left(P'\begin{pmatrix}u_0\\l_0\end{pmatrix}\right)^{-1}\right\| \le \eta_0 \qquad . \qquad (17)$$

It can be shown that the second derivative of the operator exists and is limited by the norm: $P\begin{pmatrix}u\\l\end{pmatrix}$

$$\left\|P''\begin{pmatrix}u\\l\end{pmatrix}\right\| \le \eta_2 \qquad . \qquad (18)$$

As follows from the estimates (15), (16) and (17), there is such that there is inequality for everyone $T_0 \in (t_0, T_1]$ $t \in (t_0, T_0]$

$h_0 = \eta_0{}^2 \cdot \eta_1 \cdot \eta_2 < \frac{1}{2}$.

Then, according to the above theorem [5], it follows that equation (14) has a single solution in the sphere,

$$\left\|\begin{pmatrix}u\\l\end{pmatrix} - \begin{pmatrix}u_0\\l_0\end{pmatrix}\right\| \le r < r_1 \; r_1 = \frac{1+\sqrt{1-2h_0}}{h_0}\eta_0\begin{pmatrix}u^*\\l^*\end{pmatrix}.$$

Remark. The solution of problem (5)–(8) in a domain is explicitly written out using the Green's function for the specified domain, unless the solution of the specified problem in the domain is known. $l(t) < x < \infty$ $0 < x < l(t)$

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