Proving The Inequalities Using a Definite Integral and Series

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Abstract. Definite integral and its applications are one of the basic concepts of mathematical analysis is the most powerful tool in mathematics, physics, mechanics and other subjects

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Approximate calculating definite integral and its applications are one of the basic concepts of mathematical analysis is the most powerful tool in mathematics, physics, mechanics and other subjects. We use the concept of integral widely to calculate bounded surfaces with curves, arcs and lengths of curve, volumes and various examples of proving. The definite integral has following properties

$$1^\circ.\ \int_a^b f(x)dx = -\int_b^a f(x)dx.$$

$$2^\circ.\ \int_a^b \left(f(x) \pm g(x)\right)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx.$$

$$3^\circ.\ \int_a^b k \cdot f(x)dx = k\int_a^b f(x)dx \qquad k = const.$$

$$4^\circ.\ \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, \qquad \mathrm{a < c < b}.$$

The following features of a definite integral can be successfully used to prove inequalities.

5°. If the $f(x)$ function is integrated in $[a,b]$ interval as well as non-positive and $a < b$, in that case

$$\int_a^b f(x)dx \ge 0.$$

6°. If $f(x)$ and $g(x)$ functions are integrated in $[a,b]$ interval and satisfying inequality that $f(x) \le g(x)$, *b* is larger than *a* $(a < b)$, in that case

$$\int_a^b f(x)dx \le \int_a^b g(x)dx.$$

7°. If $f(x)$ function is integrated in $[a,b]$, $|f(x)|$ function is integrated in that interval too and *b* is larger than *a* $(a < b)$, in that case

$$\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$$

inequality is appropriate.

We prove by applying those properties of a definite integral some inequalities.

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1. Prove that inequality $e^x \geq 1+x, \quad x \geq 0$.

It is known that, $e^t \geq 1$ inequality is appropriate for $\forall t \geq 0$, then in accordance with property of 6°, it is derived from

$$\int_0^x e^t dt \geq \int_0^x 1dt, \qquad x \geq 0.$$

From this we obtain $e^x - 1 \geq x$ or $e^x \geq 1+x$ inequalities by calculating the integral.

2. Prove that inequality $\arcsin x \geq x, \quad 0 \leq x < 1$.

It is known that, $\frac{1}{\sqrt{1-t^2}} \geq 1$ inequality is appropriate for number $\forall t \in (0,1)$. According to the above properties we obtain the following inequality.

$$\int_0^x \frac{1}{\sqrt{1-x^2}} dt \geq \int_0^x 1dt$$

by calculating this inequality we assure that the expressions above is true.

3. Prove that inequality $(1+x)^m \geq 1+m\cdot x, \quad x \geq 0, m \in N$

It is known that, $m(1+t)^{m-1} \geq m$ inequality is appropriate for $\forall t \geq 0$ and $\forall m \in N$. From this, for $\forall x \geq 0$

$$\int_0^x m(1+t)^{m-1} dt \geq \int_0^x mdt$$

the inequality is reasonable. Calculating the final integral, we assure the $(1+x)^m - 1 \geq m\cdot x$ or $(1+x)^m \geq 1+m\cdot x$ inequality is appropriate for $\forall x \geq 0$.

4. Prove that inequality $2\sqrt{x} > 3 - \frac{1}{x}, \qquad x \geq 1$.

We will select the corresponding $f(x)$ and $g(x)$ functions to prove that inequality. We choose $f(x) = \frac{1}{\sqrt{t}}$ and $g(x) = \frac{1}{t^2}$ functions for $\forall t \geq 1$, it is obviously, $f(x) > g(x)$ that is $\frac{1}{\sqrt{t}} > \frac{1}{t^2}$. From that

$$\int_1^x \frac{1}{\sqrt{t}} dt \geq \int_1^x \frac{1}{t^2} dt$$

inequality is appropriate for $\forall x \geq 1$. By calculating that integral, we admit that

$2\sqrt{x} > 3 - \frac{1}{x}, \ x \geq 1$ is true.

The inequality which is difficult to prove can be proved easily by series. In this case, it is used widely in the array of elementary functions as a degree of series.

Variable sign series can generally be written as follows.

$$C_1 - C_2 + C_3 - C_4 + \dots.. + (-1)^{n-1} C_n + \dots \quad (C_n > 0)$$

Theorem (Leybnis). If the ranges of the variable sign series are monotonous in absolute value and if zero is reached,

$C_{n+1} < C_{n,} \quad n = 1,2,3\dots.$ and $\lim_{n\to\infty} C_n = 0$

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then the series is approaching.

Result. If the variable sign series satisfies the theorem of Leybnis, then the residue

$$r_n = (-1)^n \left(C_{n+1} - C_{n+2} + \dots \right)$$

absolute value is less than C_{n+1}.

The following functional series is given us.

$$\sum_{n=0}^{\infty} a_n \left(z - z_0 \right)^n = a_0 + a_1 \left(z - z_0 \right) + a_2 \left(z - z_0 \right)^2 + \dots + a_n \left(z - z_0 \right)^n + \dots \qquad (1)$$

This series is called degree series. It is a private case of a functional series. If we mark that $z - z_0 = x$, then the degree series in (1) view will look like this.

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots \qquad (2)$$

So, it is possible to write any degree series in (1) is the visible series in (2). There is a radius of approximation for any degree series, there is such a possibility $r \geq 0$, that $|x| < r$ is a series of approximations for all x, $|x| > r$ is a series of distribution for satisfying all x.

If r is the approximation radius of the degree series, $(-r; r)$ becomes its approximation interval.

Theorem: degree series can be differentiated and integrated within the range of approximation.

The level of elementary functions is expressed in series as follows.

1. $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots; \quad x \in (-\infty; +\infty)$

2. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} + \dots; \quad x \in (-\infty; +\infty)$

3. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots; \quad x \in (-\infty; +\infty)$

4. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{n-1} \frac{x^n}{n} + \dots; \quad x \in (-1; 1)$

5. $\arcsin x = x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{x^7}{7} + \dots + \frac{(2n-1)!! x^{2n-1}}{2\mathrm{n}!! (2n+1)} + \dots; \ x \in (-1; 1)$

6. $arctgx = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + (-1)^{n-1} \frac{x^{2n-1}}{2n-1} + \dots; \quad x \in (-1; 1)$

7. $(1+x)^\alpha = 1 + \frac{\alpha}{1!} x + \frac{\alpha(\alpha-1)}{2!} x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} x^3 + \dots +$

$+ \frac{\alpha(\alpha-1)(\alpha-2)\dots[\alpha-(n-1)]}{n!} x^n + \dots; \quad x \in (-1; 1)$

The last series is called binomial series, here is α -optional true number.

It is possible to prove inequalities with the formulas of the degree series, theorems and concepts.

Here are some examples below.

1. For $\forall x \geq 0$ and $\forall n \in N$, prove that inequality .

$$e^x \geq 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}$$

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We know, for $\forall x \in (-\infty;+\infty)$, $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots$ equality is suitable. If $x \geq 0$, its right side is a positive series. We will go to the inequality by abandoning the after down of his n.

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots \geq 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!}$$

So, for $\forall x \geq 0$, $e^x \geq 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!}$ is true.

2. For $\forall x \in (-1,1)$, prove that inequality $\frac{1}{\sqrt{1+x^4}} \geq 1 - \frac{1}{2}x^4 + \frac{1\cdot3}{2\cdot4}x^8 - \frac{1\cdot3\cdot5}{2\cdot4\cdot6}x^{12}$.

If $\alpha = -\frac{1}{2}$ in the binomial series, for $\forall t \in (-1,1)$

$$(1+t)^{-\frac{1}{2}} = 1 - \frac{1}{2}t + \frac{1\cdot3}{2\cdot4}t^2 - \frac{1\cdot3\cdot5}{2\cdot4\cdot6}t^3 + \ldots.$$

Here $t = x^4$, $x \in (-1;1)$,

$$(1+x^4)^{-\frac{1}{2}} = 1 - \frac{1}{2}x^4 + \frac{1\cdot3}{2\cdot4}x^8 - \frac{1\cdot3\cdot5}{2\cdot4\cdot6}x^{12} + \ldots ;$$

We will have a variable series. This series is satisfied of the Leybnis theorem in $(-1;1)$ interval. So that, the following inequality is appropriate:

$$(1+x^4)^{-\frac{1}{2}} = 1 - \frac{1}{2}x^4 + \frac{1\cdot3}{2\cdot4}x^8 - \frac{1\cdot3\cdot5}{2\cdot4\cdot6}x^{12} + \ldots \geq 1 - \frac{1}{2}x^4 + \frac{1\cdot3}{2\cdot4}x^8 - \frac{1\cdot3\cdot5}{2\cdot4\cdot6}x^{12}$$

thus, for $\forall x \in (-1,1)$, $\frac{1}{\sqrt{1+x^4}} \geq 1 - \frac{1}{2}x^4 + \frac{1\cdot3}{2\cdot4}x^8 - \frac{1\cdot3\cdot5}{2\cdot4\cdot6}x^{12}; \quad -1 < x < 1;$ is suitable.

3. prove that inequality $arctgx \geq x - \frac{x^3}{3}; \quad 0 \leq x < 1.$

We know, for, $x \in (-1;1)$,

$$arctgx = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + \ldots$$

inequality is true.

Its right side is a variable series. So that, if we take its first double limit of that, we have the inequality as follows:

$$arctgx \geq x - \frac{x^3}{3}$$

Inequality is proved.

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