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Algebra of Quaternions

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Annotation: The conclusions of the Algebra and Number Theory course "Algebra of Quaternions" are highlighted as a result of a perfect and careful study of the topic, in the process of studying this topic, getting more detailed information about polynomials and information about the use of the quaternion method in algebraic examples. information provided.

Keywords: Natural numbers, integers, rational numbers, real numbers, complex numbers, abstract units, matrix, determinant, vector

Quaternions. Representation of abstract units.

Before covering this topic, we will give a brief information about the founder of the quaternion number. Irish mathematician and mechanic William Rowan Hamilton was born on August 4, 1805 in Dublin. From 1837 he was the president of the Irish FA, from 1827 he was the professor and director of the astronomical observatory of the University of Dublin. He gave a clear formal statement of the theory of complex numbers. In 1843, he created a unique system of numbers - a system of four units (l,i,j,k) called quaternions. The theory of this system made a significant contribution to the development of vector analysis. He applied the mechanical variation method (the principle of least effect) to mechanics. This principle is considered one of the main tools in generating differential equations of mechanical and physical processes. The mathematician died on September 2, 1865 in Dansink.

Natural numbers:	1,2,3,....
Integers:	-1,0,1,...
Rational numbers:	$\frac{1}{3},-\frac{2}{5},...$
Actual numbers:	$1,-1,\frac{1}{2},\pi,\sqrt{2},...$
Komplex numbers:	$3i+2,e^{i\pi/3},-2i+5,...$
Quaternions:	$1,i,j,k,\pi j-\frac{1}{2}k,...$

Quaternions.

Here, the classical notation introduced by Hamilton for quaternions is used.

Description. Quaternions number or simply quaternion (in Cartesian form) is a mathematical object of the following form:

$q=a+bi+cj+dk,$

Here a, b, c, d are real numbers and at the same time a multiplier of a real number (usually omitted in writing), i, j, k are abstract quaternion units; their squares are equal to one taken with a real negative sign.

$$i^2=j^2=k^2=-1$$

The cross-multiplication rule for these four quaternion units is defined as:

$$1j=j1=j \quad 1i=i1=i \quad 1k=k1=k$$

$$ij=-ji=k \quad jk=-kj=i \quad ki=-ik=j$$

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It can be seen that multiplication involving real units is commutative, and multiplication with abstract units is anticommutative. Thus, the complete table of quaternion multiplication of units (with $i^2 = 1$) is written in the form of 16 equalities. A compact view of this table is given in the following section. The multiplier of real units is called the scalar part of q quaternion.

$a = scalq$

A linear combination with abstract units is called a vector part.

$vectq = bi + cj + dk$

$q = a + bi + cj + dk$ all numbers in the form form a set of quaternions $Q : q \in Q$.

Sometimes quaternions are introduced through a process called double complex number system. Therefore, quaternions are called hypercomplex numbers, especially with other number systems with the following doubling order.

Representation of abstract units.

Abstract units in the algebra of quaternions, like abstract units in the algebra of complex numbers, take the form of 2×2 - matrix in algebra, where the diagonal unit matrix - E is the real unit.

If from two matrices with arbitrary components a,b,c,d,e,fm

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \qquad B = \begin{pmatrix} d & e \\ f & -d \end{pmatrix}$$

if an abstract unit is formed

$$i = \frac{A}{\sqrt{\det A}} \Rightarrow i^2 = -E, \qquad j = \frac{B}{\sqrt{\det B}} \Rightarrow j^2 = -E,$$

multiplication of them

$$ij = \frac{AB}{\sqrt{\det A \det B}} = \frac{1}{\sqrt{\det(AB)}} \begin{pmatrix} ad + bf & ae - bd \\ cd - af & ec + ad \end{pmatrix}$$

Is also "abstract unity", but under the condition that the trace of the multiplication matrix is equal to zero

$2ad + bf + ec = 0.$

It is not difficult to check that the matrices found in this way satisfy the multiplication table of quaternion units by unit E if expressed by ij=k A simple example of a triad of such matrices is proportional to the well-known Paul matrix (with an -i multiplier).

$$i = -i\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad j = -i\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, \qquad k = -i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

A different representation of quaternion units can also be presented.

In particular, the final terms of the abstract unit

$$i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

matrix substitution leads to abstract quaternion units - 4×4 represented by all real components of matrices. In order to study the properties of quaternions, it is not necessary to see the exact expression, but, for example, when defining operations on quaternions, it is usually considered that all of them are built on one quaternion unit, that is, one unit is found for all visible numbers.

Comparison, addition and multiplication of quaternions. The operation of finding the joint.

Like complex numbers, quaternion numbers are compared only to determine their equality.

$q_1 = a + bi + cj + dk$ quaternion is equal $q_2 = e + fi + gj + hk$ quaternion is equal if the coefficients corresponding to the quaternion are equal.

$q_1 = q_2 \Leftrightarrow a = e, b = f, c = g, d = h.$

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Adding (subtracting) quaternions is done by adding (subtracting) the coefficients of each of their components:

$$q_1+q_2=a+e+(b+f)i+(c+g)j+(d+h)k,$$

$$q_1-q_2=a-e+(b-f)i+(c-g)j+(d-h)k.$$

Quaternions are multiplied like polynomials, but using the multiplication table for quaternion units given above

$$q_1q_2=(a+bi+cj+dk)(e+fi+gj+hk)=ae-bf-cg-dh+(af+be)i+(ag+ec)j+(ah+ed)k+$$
$$+(bg-fc)ij+(ch-gd)jk+(df-hb)ki=ae-bf-cg-dh+(af+be+ch-gd)i+(ag+ec+df-hb)j+$$
$$+(ah+ed+bg-fc)k.$$

It can be seen that multiplication is non-commutative, that is:

$$q_1q_2\neq q_2q_1$$

Therefore, when working with quaternions, it is worth mentioning the right and left multiplication. However, it is not difficult to check that the brackets can be placed arbitrarily with multiple multipliers involved.

$$(q_1q_2)q_3=q_1(q_2q_3),$$

that is, the operation of multiplying quaternions (like real and complex numbers) is associative. Multiplication of quaternions (for left and right multiplication) and the distributive law is fulfilled in addition operations.

$$q_1(q_2+q_3)=q_1q_2+q_1q_3;(q_2+q_3)q_1=q_2q_1+q_3q_1.$$

The operation of finding the compound of a quaternion.

The operation of finding the combination of quaternions as well as complex numbers is included. For any $q=a+bi+cj+dk$ quaternion, the following quaternion can be matched as an adjunct:

$$\overline{q}=q-bi-cj-dk.$$

By finding the quaternion compound, its scalar and vector parts can be separated

$$scalq=\frac{q+\overline{q}}{2}; \qquad vectq=\frac{q-\overline{q}}{2}.$$

For joint quaternions, the following equations are valid:

$$\overline{q_1+q_2}=\overline{q_1}+\overline{q_2}; \qquad \overline{q_1q_2}=\overline{q_1}\,\overline{q_2}.$$

The last equation has the reverse order of multipliers;

In particular, due to this, the algebra of quaternions can be reduced to "four square elements".

A quaternion multiplied by its complex is a real number; The arithmetic square root of this number is called the quaternion module.

$$|q|=\sqrt{q\overline{q}}=\sqrt{a^2+b^2+c^2+d^2}.$$

The square of the quaternion modulus is called its norm.

$$q\overline{q}=|q|^2;$$

from here the formula for finding the inverse quaternion to the given quaternion can be derived

$$q^{-1}=\frac{\overline{q}}{|q|^2}.$$

If q,q_1,q_2 is given in the form of a product of multipliers, then from the definition of the norm

$$|q|^2=|q_1q_2|^2=(q_1q_2)(\overline{q_1q_2})=q_1q_2\overline{q_2}\,\overline{q_1}=q_1\overline{q_1}q_2\overline{q_2}=|q_1|^2|q_2|^2,$$

that is, the modulus of the product of quaternions is equal to the product of the modules of multipliers.

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By extension, these known quaternions are "the sum of four squares." If

$$q_1 = a + bi + cj + dk\,, \qquad q_1 = a + bi + cj + dk\,.$$

If, so

$$(ae - bf - cg - dh)^2 + (af + be + ch - gd)^2 + (ag + ec + df - hb)^2 +$$
$$+ (ah + ed + bg - fc)^2 = (a^2 + b^2 + c^2 + d^2)(e^2 + f^2 + g^2 + h^2)$$

As with complex numbers, they are read from right to left. The sum of four squares multiplied by the sum of four squares is the sum of four squares again, or the norm of the product is equal to the product of the norms.

Division of quaternions.

Due to the non-commutativity of quaternion multiplication, the devision of q_1 and $q_2 \neq 0$ can be expressed as a solution of left multiplication $q_1 = y_1 q_2$ and right multiplication $q_1 = q_2 y_1$ equations.In this case, the left division

$$y_1 = \frac{q_1 \overline{q_2}}{q_2 q_2} = \frac{q_1 \overline{q_2}}{|q_2|^2}\,,$$

right devision

$$y_2 = \frac{\overline{q_2} q_1}{q_2 q_2} = \frac{\overline{q_2} q_1}{|q_2|^2}$$

takes the appearance.

Each division has a unique expression and is a quaternion. These subdivisions are represented in an expanded view $q_1 = a + bi + cj + dk$ and for $q_2 = e + fi + gi + hk$

Algebra of quaternions and its applications.

The set of quaternions has all the properties of a four-dimensional algebra (depending on the number of base units), it defines the operations of multiplication by a real number, addition and multiplication of quaternions.

This algebra is close to the algebra of many-sided real and complex numbers. this algebra has an associative, distributive, unitary element (a real number) in relation to operations of addition and multiplication; subtraction and division are defined in it, and four square operations are performed in it. However, the algebra of quaternions is very different from the algebras of small dimensions. It is not commutative with respect to the operation of multiplication, therefore, despite its high-order properties, the set of quaternions does not form a field, but is considered an object with a divisional noncommutative ring. In some cases, the name "non-commutative field" can also be found. One of the important features that distinguish the algebra of quaternions is that it is the last of the dimensions of the associative algebra with unit and division. In 1878, the German mathematician G. Frobenius proved the following amazing theorem: "Any partitioned associative algebra is isomorphic to one of three algebras: the algebra of real numbers, the algebra of complex numbers, and the algebra of quaternions." the associative multiplication with unity is lost, and analog - alternation takes its place. It was the search for such a phenomenon that led the English mathematician A. Kelini to discover the algebra of octaves. It turned out that there are no algebras of larger dimensions that satisfy the sum of squares clause. This was proved in 1898 by the mathematician A. Hurvitsen through the following theorem: "Any normalized algebra is isomorphic to the following algebras: real numbers, complex numbers, quaternions, and octaves."

The last four algebras, whose dimensions can be determined using exponential series with 2 bases:

Algebra	Al algebraic size $n = 2^p$	p Index
H Real numbers	$1 = 2^0$	0 0

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Complex nubmers	$2 = 2^1$	1 1
Quaternions	$4 = 2^2$	2 2
Oktavas	$8 = 2^3$	3 3

Another formal situation can be mentioned. Algebra of quaternions can be seen as a special case of the general mathematical construction - Clifford's algebra. the latter is made up of natural numbers that form units, the half sum of even products is a unit matrix of the order of algebraic dimension. Based on this point of view, the algebra of quaternions is a two-dimensional Clifford algebra. Here, two arbitrary abstract entities are taken as constituents; the remaining units are defined as arbitrary multiples of the generators. In the literature, quaternion algebra is defined by or (the validity of the coefficients in the root of quaternion units is emphasized). Sometimes marked with Q.

Geometric application of quaternions.

A geometric image that satisfies all quaternion sets has not yet been determined. The number of dimensions of quaternion algebras and specifying one of these dimensions, the coefficients in front of the quaternion units, seem to encourage us to understand the coordinates of points in the physical space-time continuum. However, such a linear solution would be hasty. It seems that it is not so general that the Cartesian axis of the unit in every quaternion algebra is related to a certain direction (as in complex number expressions).

The geometric representation of original vector quaternions is much easier. For this, it is enough to know the following, that is, in the multiplication of two vectors: $p = x_1 i + x_2 j + x_3 k$ and quaternion $q = y_1 i + y_2 j + y_3 k$

$$pq = (x_1 i + x_2 j + x_3 k)(y_1 i + y_2 j + y_3 k) = -(x_1 y_1 + x_2 y_2 + x_3 y_3) + (x_2 y_3 - x_3 y_2) i +$$
$$+ (x_3 y_1 - x_1 y_3) j + (x_2 y_3 - x_3 y_2) k$$

The scalar part contains an expression similar to the scalar product of two vectors in Cartesian coordinates, and the vector part contains an expression similar to the vector product.

It is concluded that:

Abstract quaternion units define the direction of the right Cartesian coordinate system, and the coefficients before them are the components of the vector in this coordinate system.

This explanation given by W. Hamilton to vector quarnions was used by Maxwell in writing his electrodynamic equation.

Conclusion

This article is prepared on the topic "Algebra of quaternions" and consists of an introduction, main body, conclusion and a list of references. In the process of preparing this article, this science, i.e. Algebra and quaternion algebra of number theory, was described in detail, which includes the transcendental extension of the whole field. Theorems about these are also proved and examples are given.

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